Serre Duality

We will look at Serre Duality in the case of toric varieties and prove the statement for toric varieties.

We begin by stating the general theorem.

Theorem 1. Let X be a projective smooth variety and E a vector bundle over X. Then

$$H^i(X,E)^* = H^{n-i}(X, E^\vee \otimes \Omega_X^n)$$

where Ω_X^n is the canonical line bundle (or top-dimensional forms).

We will be interested in cases where X is a complete toric variety and $E = \mathcal{O}(D)$ is a line bundle for some T-Divisor D. We saw in the previous talk that then $\Omega_X^n = \mathcal{O}(-\sum D_i)$.

Let us first look at an example in the case of toric varieties.

Example. We can use Serre Duality to calculate $H^2(\mathbb{P}^2, \mathcal{O}(-4))$. Since $\text{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$, we have $\Omega_X^n = \mathcal{O}(-\sum D_i) \cong \mathcal{O}(-3)$. Serre Duality gives

$$H^2(\mathbb{P}^2, \mathcal{O}(-4))^* \cong H^0(\mathbb{P}^2, \mathcal{O}(4) \otimes \mathcal{O}(-3)) \cong H^0(\mathbb{P}^2, \mathcal{O}(1))$$

We can calculate $H^0(\mathbb{P}^2, \mathcal{O}(1))$ using a piecewise-linear function and deduce that $H^2(\mathbb{P}^2, \mathcal{O}(-4))$ is three-dimensional.

Let X be a toric variety and $E = \mathcal{O}(D)$ a line bundle for some T-Divisor D. Remember that we have a grading

$$H^i(X, \mathcal{O}(D)) = \bigoplus_{u \in M} H^i(X, \mathcal{O}(D))_u$$

Furthermore we can calculate $H^i(X, \mathcal{O}(D))_u$ using local cohomology groups

$$H^i(X, \mathcal{O}(D))_u = H^i_{Z(u)}(|\triangle|) \tag{1}$$

where $H^i_{Z(u)}(|\triangle|) = H^i(|\triangle|, |\triangle| - Z(u); \mathbb{C})$ and

$$Z(u) = \{v \in |\triangle| \mid \langle u, v\rangle \geq \psi(v)\}$$

where $\psi = \psi_D$ is the piecewise-linear function corresponding to D.

Let us restate Serre Duality for toric varieties.

Theorem 1' Let $X = X(\triangle)$ be a complete toric variety and $E = \mathcal{O}(D)$ a line bundle over X. Then

$$H^i(X, \mathcal{O}(D))^* = H^{n-i}(X, \mathcal{O}(-D) \otimes \mathcal{O}(-\sum D_i)).$$

Furthermore the isomorphism respects the grading, i.e.

$$H^i(X, \mathcal{O}(D))^*_u = H^{n-i}(X, \mathcal{O}(-D) \otimes \mathcal{O}(-\sum D_i))_{-u}.$$

Using (1) the proof amounts to showing

$$H^i_{Z(u)}(N_\mathbb{R})^* = H^{n-i}_{Z'(-u)}(N_\mathbb{R})$$

for all $u \in M$ where

$$\begin{aligned} Z(u) &= \{v \in N_{\mathbb{R}} \mid \langle u, v\rangle \geq \psi(v)\} \\ Z'(-u) &= \{v \in N_{\mathbb{R}} \mid \langle -u, v\rangle \geq -\psi(v) + \kappa(v)\} \\ &= \{v \in N_{\mathbb{R}} \mid \langle u, v\rangle \leq \psi(v) - \kappa(v)\} \end{aligned}$$

where $\psi = \psi_D$, $\kappa = \psi_{-\sum D_i}$ are the piecewise-linear functions corresponding to D, $-\sum D_i$. Also notice that $\kappa(v_i) = 1$.

Fix $u \in M$ and let $Z = Z(u)$, $Z' = Z'(-u)$. Let

$$S = \{v \in N_{\mathbb{R}} \mid \kappa(v) = 1\},$$

the boundary of a polyhedral ball B, so it is a deformation retract of $N_{\mathbb{R}} - \{0\}$.

The proof will be almost purely topological. We need a few lemmas first.

Lemma 1. If C is a nonempty closed cone in $N_{\mathbb{R}}$, there are isomorphisms

$$H^i_C(N_{\mathbb{R}}) = H^i(N_{\mathbb{R}}, N_{\mathbb{R}} - C) \overset{(1)}{\cong} H^i(B, S - (S \cap C)) \overset{(2)}{\cong} \tilde{H}^{i-1}(S - (S \cap C)) \overset{(3)}{\cong} \tilde{H}_{n-i-1}(S \cap C)$$

Proof. The very first equality is just the definition of $H^i_C(N_{\mathbb{R}})$.

(1) follows from the fact that there is a deformation retraction such that $N_{\mathbb{R}}$ deformation retracts onto B and $N_{\mathbb{R}} - C$ deformation retracts onto $S - (S \cap C)$.

(2) follows from the LES of of the reduced cohomology groups of the pair $(B, S - (S \cap C))$, in particular B is contractible so we get

$$\cdots \to \tilde{H}^{i-1}(B) \to \tilde{H}^{i-1}(S - (S \cap C)) \overset{\cong}{\to} H^i(B, S - (S \cap C)) \to \tilde{H}^i(B) \to \ldots$$

(3) this follows from Alexander Duality ☐

Lemma 2. The embedding $S \cap Z \hookrightarrow S - (S \cap Z')$ is a deformation retract.

Proof. We will describe a homotopy on each cone σ of our fan (remember that our toric variety is complete).

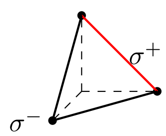
$$\begin{aligned} (\sigma \cap S) \cap Z &= \{v \in \sigma \cap S \mid \langle u, v\rangle \geq \psi(v)\} \\ (\sigma \cap S) \cap Z' &= \{v \in \sigma \cap S \mid \langle u, v\rangle \leq \psi(v) - 1\} \\ (\sigma \cap S) - ((\sigma \cap S) \cap Z') &= \{v \in \sigma \cap S \mid \langle u, v\rangle > \psi(v) - 1\} \end{aligned}$$

Notice that $(\sigma \cap S) \cap Z$ and $(\sigma \cap S) \cap Z'$ are disjoint. Let v_i be the lattice points on the different edges of our cone. Let

$$\begin{aligned} \sigma^+ &= \operatorname{Conv}(v_i \mid v_i \in (\sigma \cap S) \cap Z) \\ \sigma^- &= \operatorname{Conv}(v_i \mid v_i \in (\sigma \cap S) \cap Z') \end{aligned}$$

Then we can write every $v \in (\sigma \cap S)$ uniquely as

$$v = av^+ + bv^-, \text{ where } v^+ \in \sigma^+,\ v^- \in \sigma^-,\ a,\ b > 0,\ a + b = 1.$$



Then our homotopy for $v \in (\sigma \cap S) - ((\sigma \cap S) \cap Z')$ will look like

$$\phi_t(v) = \begin{cases} tv^+ + (1-t)v & \text{if } v \notin Z,\ \langle u, tv^+ + (1-t)v\rangle \leq \psi(tv^+ + (1-t)v) \\ t_0v^+ + (1-t_0)v & \text{if } v \notin Z,\ \langle u, t_0v^+ + (1-t_0)v\rangle = \psi(t_0v^+ + (1-t_0)v),\ t > t_0 \\ v & \text{if } v \in (\sigma \cap S) \cap Z) \end{cases}$$

Now gluing together the homotopies will show the statement. □

We can now prove our version of Serre Duality.

Proof. There are isomorphisms

$$H^{n-i}_{Z'}(N_{\mathbb{R}}) \overset{(1)}{\cong} \tilde{H}^{n-i-1}(S - (S \cap Z')) \overset{(2)}{\cong} \tilde{H}_{n-i-1}(S - (S \cap Z'))^* \overset{(3)}{\cong} \tilde{H}_{n-i-1}(S \cap Z)^* \overset{(4)}{\cong} H^i_Z(N_{\mathbb{R}})^*$$

(1) follows from Lemma 1, with $C = Z'$

(2) this is the universal coefficient theorem for cohomology groups, note that we have torsion-free coefficients because we take coefficients in $\mathbb{C}$

(3) by Lemma 2 $S \cap Z$ is a deformation retract of $S - (S \cap Z')$

(4) this is again Lemma 1 with $C = Z$ □

Remark 2. The statement

$$H^i_{Z(u)}(N_{\mathbb{R}})^* = H^{n-i}_{Z'(-u)}(N_{\mathbb{R}}) \tag{2}$$

is quite trivial for points $u \in M$ where $Z(u) = N_{\mathbb{R}}$ and $Z'(-u) = \{0\}$ or the other way around. In this particular case, (2) is just

$$H^i(N_{\mathbb{R}}, \emptyset)^* = H^i(N_{\mathbb{R}})^* \cong \tilde{H}^{n-i}(S^n) = H^{n-i}(N_{\mathbb{R}}, N_{\mathbb{R}} - \{0\})$$

or the other way around.