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Resolution in Toric Varieties

Seminar in Toric Geometry

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Contents

Chapter 1

Goal of the session

The goal of this session is the following : We are given a fan Δ with corresponding toric variety $X_\Delta = X(\Delta, N)$ (we won't change the lattice unexpectedly) in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and we try to find a resolution of singularities which is the data of

1. A smooth variety Y
2. A proper birational morphism $f : Y \to X_\Delta$ so that f induces an isomorphism to the complement of the singular points

$$f^{-1}\left(X_\Delta \setminus (X_\Delta)_{sing}\right) \xrightarrow{\sim} X_\Delta \setminus (X_\Delta)_{sing}.$$

We recall some important concepts.

Remark 1.1 *1. Any strongly convex rational polyhedral cone has up to multiplication with $\mathbb{R}_{>0}$ a unique set of minimal generators : the so-called ray generators. We will always choose them to have coprime coordinates along N; so they correspond exactly to the rays (one dimensional faces) of the cone.*

2. A cone is simplicial if its ray generators are linearly independent.

3. A cone is smooth if its ray generators can be extended to a basis of N.

We will use the following strategy to transform a fan. The idea is to make our fan smooth, because in the previous session we have seen that smoothness of the fan translates to smoothness of the toric variety.

Definition 1.2 (star subdivision) *Let Δ be a fan in $N_\mathbb{R}$ and $\rho = \mathrm{cone}(v)$ for some primitive $v \in N_\mathbb{R}$. We then define the star subdivision $\Sigma^*_\rho(\Delta)$ of Δ with center ρ to be the collection*

$$\{\sigma \in \Delta | \rho \not\subset \sigma\} \cup \{\rho + \tau | \tau \in \Delta, v \notin \tau \text{ and } \exists \sigma \in \Delta \text{ with } \rho + \tau \subset \sigma\} \quad (1.1)$$

Proposition 1.3 *Let $\Delta \subset N_{\mathbb{R}}$ be a fan in a lattice N and $\rho \subset |\Delta| \cap N$ a ray, and $\Sigma = \Sigma^*_\rho(\Delta)$. If Σ contains all the smooth cones of Δ, then the induced morphism $f : X_\Sigma \to X_\Delta$ is proper and birational.*
More precisely, there is an induced isomorphism

$$f^{-1}\left(X_\Delta \setminus (X_\Delta)_{sing}\right) \xrightarrow{\sim} X_\Delta \setminus (X_\Delta)_{sing}$$

where the two former subsets are open. This result generalizes to the situation where Σ is obtained by a finite sequence of star refinements from Δ.

Proof See for instance [4] proposition 6.11. □

We have now reduced our task to finding a smooth fan by only using the above technique.

Proposition 1.4 *Let σ be a strongly convex rational polyhedral cone that has at least 1 generator. Then σ is simplicial if and only if any face of σ is split.*

Proof See [4] proposition 6.9. □

Proposition 1.5 *For any fan Δ there is a refinement Δ' of Δ that is simplicial. Furthermore, one can choose Δ' that contains all simplicial cones of Δ.*

Proof Our proof is taken from [4] Proposition 6.10. We will prove the following statement by induction on k.
Let $k \in \mathbb{N}$, and Δ a fan. Then, there is a finite sequence of star refinements with resulting fan Δ' such that all cones that have at most k minimal generators are simplicial. Moreover, $\{\sigma \in \Delta | \sigma \text{ } simplicial\ \} \subset \Delta'$.

If $k \leq 2$ then $\Delta = \Delta'$ works, so we may assume $k > 2$ and all cones with at most $k-1$ minimal generators are simplicial (we have applied a finite sequence of star refinements).

We set

$$A_\Delta(k) = \{\sigma \in \Delta | \sigma \textit{ has k minimal generators and is stout }\}$$

and for all $\gamma \in A_\Delta(k)$ we pick a ray ρ that is a face of γ and we perform $\Sigma^*_\rho(\Delta)$. Now, inside γ there are no stout faces anymore. Let $\beta \in \Sigma^*_\rho(\Delta)$ be a face of γ. Then there are two cases

1. $\rho \not\subset \beta$. Then $\beta \in \Delta$ has strictly less ray generators than γ so it is simplicial so it can be split.
2. $\beta = \tau + \rho \subset \gamma$ for $\tau \in \Sigma^*_\rho(\Delta)$. Note that τ was in the previous fan but it does not intersect ρ. $\tau = \tau \cap \beta$ is a face of each so τ is a face of β. So β is split.

It follows that inside the support of γ there are no stout faces. We observe that

$$A_{\Sigma^*_\rho(\Delta)}(k+1) = A_\Delta(k+1). \tag{1.2}$$

$$A_{\Sigma^*_\rho(\Delta)}(k) \subsetneq A_\Delta(k). \tag{1.3}$$

Any newly introduced cone is split, so in order to be stout one has to come from the old fan. This implies the above claims.

We choose an enumeration of this set so that $A_\Delta(k) = \{\gamma_1, \dots, \gamma_r\}$. Let $\Delta_0 = \Delta$ and for all $i \leq r$ we pick an edge ρ_i of γ_i if γ_i is still stout. We then set

$$\Delta_i = \Sigma^*_{\rho_i}(\Delta_{i-1}),$$

otherwise we just keep $\Delta_i = \Delta_{i-1}$ and we claim that $\Delta_r = \Delta'$.
We apply the above arguments on Δ_{i-1} instead of Δ. We know that in the support of γ_i there are no more stout faces. If on the other hand, ε is a new face with k minimal generators in the fan Δ_i then by construction $\varepsilon = \tau + \rho_i$ where $\tau \not\subset \gamma_i$. So ε is split and if α is any strict face of ε then either $\alpha \in \Delta_{i-1}$ or $\alpha = \tau_\alpha + \rho_i$ for $\tau_\alpha \in \Delta_i$ (as discussed before). In the first case, it is simplicial, because it has less than k ray generators, hence it is split; in the second case it is obviously split. This shows that none of these refinements creates new stout faces with k minimal generators. More precisely by 1.3

$$A_{\Delta_i}(k) \subset \{\gamma_{i+1}, \dots, \gamma_r\}.$$

Hence

$$A_{\Delta_r}(k) \subset \varnothing$$

so we deduce that all cones with at most k ray generators of the last fan are simplicial (all its faces are split).

An important remark is that $A_\Delta(k+1)$ is finite. But this follows from 1.2. In fact, $A_{\Delta_i}(k+1)$ is finite for all i.
The last step is to show that $\{\sigma \in \Delta | \sigma \textit{ simplicial}\} \subset \Delta'$. If $\sigma \in \Delta_{i-1}$ is simplicial with $\rho_i \subset \sigma$, then by [4] Lemma 6.8 (iv), $\sigma = \tau + \rho_i$ for a strict face τ of σ hence $\tau \in \Delta_{i-1}$. Such an object is in Δ_i by construction. This proves the inclusion.
Put together, these claims imply the proposition, because every cone in the fan has finitely many ray generators. □

Chapter 2

Multiplicities of simplicial cones

In this chapter, we look at a strongly convex rational polyhedral cone σ that is simplicial, and we make it smooth. In fact, not exactly; we replace it by a smooth fan with same support. In the previous chapter we produced simplicial cones from arbitrary cones, so this might finalise the resolution.

Definition 2.1 (Multiplicity) *Let σ be a simplicial strongly convex rational polyhedral cone with primitive ray generators $v_1, \dots, v_k$, so $\sigma = \operatorname{cone}(v_1, \dots, v_k)$. Let $N_\sigma = \operatorname{Span}(\sigma) \cap N$ and $G_\sigma = \sum_{j=1}^{k} \mathbb{Z} v_j \subset N$. Then, we define using the group-theoretical index,*

$$\operatorname{mult}(\sigma) = [N_\sigma : G_\sigma]. \tag{2.1}$$

As an example to illustrate that these lattices do not always coincide, we consider the cone σ in $\mathbb{R}^2$ with generators $(0,1), (2,-1)$. Then $(1,1) \in N_\sigma \setminus G_\sigma$. Indeed, σ had generators $(0,1), (2,-1)$, so $N_\sigma = N$.
We observe the following:

Lemma 2.2 $\operatorname{mult} \sigma = 1$ *if and only if σ is smooth/regular.*

Proof Let $\sigma = \operatorname{cone}(v_1, \dots, v_k) \subset N_\mathbb{R}$. If $\operatorname{mult} \sigma = 1$ then $G_\sigma = N_\sigma$. It is not hard to see that N/N_σ is torsion-free, so by A.1, there is a sublattice $H_\sigma \subset N$ with $N = N_\sigma \oplus H_\sigma = G_\sigma \oplus H_\sigma$. The proof of A.1 implies that H_σ comes with a basis B, so that $B \cup \{v_1, \dots, v_k\}$ is a basis for N. B extends the (v_i) to a basis. Hence σ is smooth.
For the other implication, let $v = \sum_{i=1}^{k} a_i v_i \in N_\sigma$, $a_i \in \mathbb{R}$. We know that we can find primitive vectors $v_{k+1}, \dots, v_n$ such that $v_1, \dots, v_n$ is a basis for N. Let n_i be the integer part of a_i in the sense that there are $b_i \in [0,1)$ with $a_i = n_i + b_i$. We set $y = \sum_{i=1}^{k} n_i v_i \in G_\sigma \subset \mathbb{N}_\sigma$ and $p = \sum_{i=1}^{k} b_i v_i \in \operatorname{Span} \sigma$. From

$$v = y + p$$

and N_σ being a group, we deduce

$$p = \sum_{i=1}^{k} b_i v_i = \sum_{i=1}^{n} \tilde{a}_i v_i \in N_\sigma \subset N$$

where $\tilde{a}_i \in \mathbb{Z}$. This happens because the v_i are a basis over $\mathbb{Z}$ for N. But since a representation in the basis is unique, we see that $b_i = 0$ for all i. From the equation that is alone in its line, we deduce that $v = y \in G_\sigma$. This shows that $G_\sigma = N_\sigma$ so we have $\operatorname{mult} \sigma = 1$. □

Definition 2.3 (parallelotope of a cone) *Let σ be a strongly convex rational polyhedral cone spanned by primitive $v_1, \dots, v_k$ in the lattice. Then, we say that*

$$P_\sigma = \left\{ \sum_{j=1}^{k} \lambda_j v_j \,\middle|\, \lambda_j \in [0, 1) \right\} \tag{2.2}$$

is the parallelotope of σ.

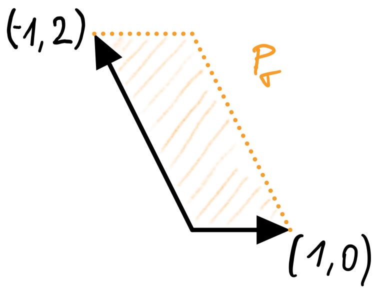

Figure 2.1: two-dimensional example

The parallelotope contains some information about multiplicity. More precisely we have

Proposition 2.4 *Let σ be a simplicial strongly convex rational polyhedral cone generated by its ray generators $v_1, \dots, v_k$ in a lattice N. Then*

i) It holds

$$\operatorname{mult} \sigma = |P_\sigma \cap N| \tag{2.3}$$

ii) N_σ is free of rank k

iii) Let $e_1, \dots, e_k$ a basis of the lattice N_σ. If we write $v_i = \sum_{j=1}^{k} a_{ij} e_j$ and consider the matrix A with coefficients a_{ij}, then

$$\operatorname{mult} \sigma = |\det A| \tag{2.4}$$

iv) mult *is non-decreasing with respect to inclusion of simplicial faces.*

Proof We follow the proof of [1]. By G_σ we denote the same group as in 2.1.

i) We have a homomorphism f that is the composition of the projection modulo subgroup and the inclusion.

$$\begin{tikzcd} P_\sigma \cap N \arrow[r, "\iota"] \arrow[dr, "f"'] & N_\sigma \arrow[d, "\pi_{G_\sigma}"] \\ & N_\sigma/G_\sigma \end{tikzcd}$$

The inclusion holds, because $P_\sigma \subset \operatorname{Span}\sigma$. We will show that f is injective. For $(\lambda_1, \ldots, \lambda_k), (\mu_1, \ldots, \mu_k) \in [0,1)^k$ we have that

$$f\left(\sum_{i=1}^{k} \lambda_i v_i\right) = f\left(\sum_{i=1}^{k} \mu_i v_i\right)$$

implies

$$\sum_{i=1}^{k} (\lambda_i - \mu_i)\, v_i \in G_\sigma \text{ meaning that } \lambda_i - \mu_i \text{ are integers}$$

because the v_i are linearly independent (i.e. the representation above is unique) which only holds if $\mu_i = \lambda_i$ for all i. So f is injective.
On the other hand, if we have an element $v = \sum_{i=1}^{k} x_i v_i + G_\sigma \in N_\sigma/G_\sigma$ then for y_i being x_i from which its integer part is substracted, we obtain that $f\left(\sum_{i=1}^{k} y_i v_i\right) = v$ and $\sum_{i=1}^{k} y_i v_i \in P_\sigma \cap N$.

ii) First, we explain why

$$\operatorname{mult}\sigma < +\infty$$

We had $\operatorname{mult}\sigma = |P_\sigma \cap N|$. In fact this is a constraint on the maximal distance of a lattice point to the origin : If $v \in P_\sigma \cap N$ then by picking the isomorohism $\mathbb{R}^n \xrightarrow{\sim} N_{\mathbb{R}}$, that identifies the basis elements of N with the standard basis of $\mathbb{R}^n$, we have that $\|v\|^2 \leq \|v_1\|^2 + \cdots + \|v_k\|^2 =: p$ which is finite. So to pick an element in $P_\sigma \cap N$ there are at most p^n choices for all coordinates, so $\operatorname{mult}\sigma \leq p^n$. A Theorem from group theory tells us that a subgroup of a finitely generated free abelian group, is free abelian. (See for instance [3] Theorem 9.60. recalling that free abelian groups are free modules over the integers). We see from $\operatorname{Span}\sigma = N_\sigma \otimes \mathbb{R}$ that N_σ is free of rank $k = \dim(\operatorname{Span}\sigma)$. In fact, the vector space V_σ generated by σ contains V_{min}, the one generated by the ray generators of σ. But σ is simplicial, so $V_\sigma = V_{min}$. By the discussion in **??** the lattices G_σ, N_σ have the same rank. The expression for $\operatorname{Span}\sigma$ that we claimed can be deduced by considering the map that is induced on the tensor product by the bilinear multiplication $N_\sigma \times \mathbb{R} \to \operatorname{Span}\sigma$.

iii) We take the proof from [3] corollary 9.63 that uses the Smith Normal Form from linear algebra. The theorem about Smith Normal Form gives matrices Q, B, P with coefficients in $\mathbb{Z}$ so that

$$A = QBP$$

and Q, P are invertible and $B = diag(g_1, \ldots, g_n)$. As $1, -1$ are the only units in $\mathbb{Z}$, we have $|\det A| = |\det B| = g_1 \ldots g_n$. Corollary 5.30 in [3] says that since the g_i are the so-called invariant factors of ${}^{N_\sigma}/_{G_\sigma}$, we have that their product is $|{}^{N_\sigma}/_{G_\sigma}|$. (For brevity's sake we refer to [3] Theorem 9.34 and Corollary 9.61 for a deeper discussion of invariant factors.)

iv) This follows from (i). □

Proposition 2.5 (multiplicity reduction—local version) *Let Δ be a simplicial fan in $N_\mathbb{R}$ and $\sigma = \operatorname{cone}(v_1, \ldots, v_k) \in \Delta$ with $\operatorname{mult} \sigma > 1$. Then there is a ray $\rho \in P_\sigma \cap N$ with (and it holds for any such ray)*

i) For all $\gamma \in \Sigma^*_\rho(\Delta)$ with $\gamma \subset \sigma$ it holds $\operatorname{mult} \gamma < \operatorname{mult} \sigma$ or $\gamma \in \Delta$

ii) $\gamma \in \Delta$ is smooth $\implies \gamma \in \Sigma^*_\rho(\Delta)$

iii) $\Sigma^*_\rho(\Delta)$ is simplicial.

Proof We follow the proofs of [2],[4].
By 2.4 (i), there is $v = \sum_{i=1}^k \lambda_i v_i \in (P_\sigma \cap N)$ where all $\lambda_i \in [0, 1)$ but not all are 0. Let $\rho = \operatorname{cone}(v)$ and $\Delta^* = \Sigma^*_\rho(\Delta)$. There is no smooth cone in Δ that has v as an element. If $\alpha \in \Delta$ was smooth containing v then $v \in (\sigma \cap \alpha) \in \Delta$ which is a face of both. As the cones are simplicial, $\sigma \cap \alpha = \operatorname{cone}(S)$ where $S \subset \{v_1, \ldots, v_k\}$ provides ray generators of $\sigma \cap \alpha$. By the same argument, α has ray generators $S \cup S'$. This means that $v \in P_\alpha \cap N$ so $\operatorname{mult} \alpha > 1$ so it would not be smooth.
If $\gamma \in \Delta$ is smooth then by the above paragraph, $\rho \not\subset \gamma$, hence $\gamma \in \Sigma^*_\rho(\Delta)$. This implies (ii).
Let $\gamma \in \Sigma^*_\rho(\Delta) \setminus \Delta$ be contained in σ. We can write $\gamma = \tau + \rho \subset \sigma$ for a face $\rho \not\subset \tau$.
We observe that γ is simplicial (this implies (iii)) so its multiplicity is well-defined. We have that $v_i \notin \tau$ for some $i \leq k$ (note that the corresponding λ_i is non-zero in the representation of v). Let $E = \{v_1, \ldots, v_n\}$. Lemma 6.8 in [4] shows that $\tau + \rho$ is a face of $\tau_i + \rho$ because $\tau_i + \rho$ is simplicial (it has ray generators $\{v\} \cup S_i$ where $S_i = E \setminus \{v_i\}$). And v cannot be in the span of τ_i, as we need v_i to get it. By 2.4

$$\operatorname{mult}(\tau + \rho) \leq \operatorname{mult}(\tau_i + \rho).$$

We have a basis $e_1, \ldots, e_k$ of N_σ and a basis $f_1, \ldots, f_k$ of $N_{\tau_i + \rho}$ and a matrix A with coefficients (a_{ij}) given by

$$v_q = \sum_{j=1}^{k} a_{qj} f_j$$

and

$$v = \sum_{q=1}^{k} \lambda_q v_q = \sum_{q,j \leq k} \lambda_q a_{qj} f_j.$$

By 2.4, $\operatorname{mult}(\tau_i + \rho)$ is up to sign equal to

$$\det \begin{pmatrix} a_{1,1} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & \sum_{q\leq k} \lambda_q a_{q1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,i-1} & a_{k,i+1} & \cdots & \sum_{q\leq k} \lambda_q a_{qk} \end{pmatrix}$$

Using the linearity in every column and the fact that if two columns are proportional then the determinant vanishes, we obtain

$$\operatorname{mult}(\tau_i + \rho) \sim \det \begin{pmatrix} a_{1,1} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & \lambda_i a_{i1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,i-1} & a_{k,i+1} & \cdots & \lambda_i a_{ik} \end{pmatrix} \sim \lambda_i \det(A)$$

up to sign.
If we write B for the matrix with coefficients satisfying $v_i = \sum_{j=1}^{k} b_{ij} e_j$, then $\operatorname{mult} \sigma = |\det(B)|$ 2.4. If we consider the homomorphisms $G_\sigma \to N_\sigma$, $G_\sigma \to N_{\tau_i \cap \rho}$ given by inclusion (and their corresponding matrices B, A) and $N_{\tau_i \cap \rho} \cong N_\sigma$ then we see that we can apply [3] Proposition 9.53 to obtain that $B = QAP$ where $\det(P), \det(Q) = \pm 1$. By multiplicativity of the determinant, $\operatorname{mult}(\tau_i + \rho) = \lambda_i |\det(B)| = \lambda_i \operatorname{mult} \sigma < \operatorname{mult} \sigma$. We deduce $\operatorname{mult} \gamma = \operatorname{mult}(\tau + \rho) < \operatorname{mult} \sigma$. $\square$

Theorem 2.6 *Let Δ be a simplicial fan. Then there is a refinement Δ' of Δ that is smooth. Moreover*

$$\{\gamma \in \Delta | \gamma \textit{ is smooth } \} \subset \Delta'.$$

Proof We briefly emphazise that multiplicities are always finite, see for instance 2.4 (iii) and we recall that if all cones in a fan have multiplicity 1 then they are smooth, so the fan is smooth. We will show the following claim via induction on $M \geq 1$.
Let Δ be a simplicial fan, with $M = \max_{\sigma \in \Delta}(\operatorname{mult} \sigma)$. Then, there is a refinement Δ' of Δ such that Δ' contains all smooth cones of Δ and

$$\max_{\sigma' \in \Delta'}(\operatorname{mult} \sigma') = 1.$$

If $M = 1$ then it holds.
If $M > 1$, we define

$$S_\Delta(l) = \{\gamma \in \Delta | \operatorname{mult} \gamma = l\}$$

and $T_\Delta(l) = |S_\Delta(l)|$. We set $T = T_\Delta(M)$. We construct a sequence of fans Δ_l as follows: $\Delta_0 = \Delta$ and we know by 2.5 that if a fan has an element with multiplicity at least 2 then we can apply a star refinement so we may define

$$\Delta_{l+1} = \begin{cases} \Sigma^*_{\rho_{l+1}}(\Delta_l) & \text{if } \Delta_l \text{ has an element with multiplicity M} \\ \Delta_l & \text{otherwise} \end{cases}$$

where we pick ρ_l according to 2.5. We state the recursion

$$T_{\Delta_l}(M) \leq T_{\Delta_{l-1}}(M) - 1.$$

Let $\sigma_0 \in \Delta_{l-1}$ with $\operatorname{mult} \sigma_0 = M$ so that Δ_l is the star refinement of Δ_{l-1} centered at a certain non-zero $\rho_l \subset \sigma_0$. We claim that any $\gamma \in \Delta_l$ was in Δ_{l-1} or has $\operatorname{mult} \gamma < M$.

If $\gamma \notin \Delta_{l-1}$ then $\gamma = \tau + \rho_l \subset \sigma$ where $\tau, \sigma \in \Delta_{l-1}$ by construction. Suppose that $\rho_l = \operatorname{cone}(v)$ and $\sigma_0 = \operatorname{cone}(S)$, where $v = \sum_{s \in S} \lambda_s s$ where for all s, $0 \leq \lambda_s < 1$. We set S' to be the set of $s \in S$ for which $\lambda_s \neq 0$ and $\sigma_1 = \operatorname{cone}(S')$ a simplicial face of σ_0 the parallelotope of which also contains v. Now, $v \in \sigma \cap \sigma_1$ implies that σ_1 is a face of σ by linear independence of the ray generators. Hence $v \in P_\sigma \cap N$. By 2.4, $1 < \operatorname{mult} \sigma_1 \leq \operatorname{mult} \sigma \leq M$. so we can apply Theorem 2.5 to the situation $\rho_l \subset \sigma$ instead of σ_0 so $\operatorname{mult} \gamma < \operatorname{mult} \sigma \leq M$.

We have achieved that $\sigma_0 \notin \Delta_l$ because it contains ρ_l and all newly introduced cones have strictly smaller multiplicities than $M = \operatorname{mult} \sigma_0$. This implies

$$S_{\Delta_l} \subset S_{\Delta_{l-1}} \setminus \{\sigma_0\}$$

which translates to the recursion. The latter implies

$$T_{\Delta_l}(M) \leq T_{\Delta_0}(M) - l = T - l$$

so that

$$T_{\Delta_T}(M) \leq T - T = 0.$$

This means that Δ_T is a refinement of Δ that contains all smooth cones of Δ by 2.5 and all cones have at most multiplicity $M - 1$. By the induction hypothesis, the theorem follows. □

Chapter 3

Conclusion

In this chapter, we just sum up some previously discussed results. One might call it a resolution of singularities in toric varieties. We recall what it means to be a resolution.

Definition 3.1 *A morhism of schemes $f : Y \to X$ is a resolution of singularities if Y is smooth, f is proper and if it induces an isomorphism $f^{-1}\left(X \setminus X_{sing}\right) \xrightarrow{\sim} X \setminus X_{sing}$.*

Theorem 3.2 (Resolution of singularities in toric varieties) *Let Δ be a fan, and X_Δ the corresponding toric variety. Then there is a fan Σ and a resolution of singularities $f : X_\Sigma \to X_\Delta$.*

Proof We may use 1.5 to obtain a simplicial refinement Γ of Δ such that

$$\{\sigma \in \Delta | \sigma \text{ is smooth}\} \subset \Gamma$$

and Theorem 2.6 to obtain a refinement Σ of Γ, hence of Δ, that is smooth and contains all smooth cones of Γ. Together

$$\{\sigma \in \Delta | \sigma \text{ is smooth}\} \subset \Sigma.$$

This produces a morphism

$$f : X_\Sigma \to X_\Delta$$

that is proper by Proposition 1.3 and birational to the complement of the singular points. So f is a resolution of singularities because X_Σ is smooth since smoothness is a local property. $\square$

Appendix A

Appendix

Lemma A.1 *Let N be a lattice and $N_1 \subset N$ such that ${}^{N}/_{N_1}$ is torsion-free. Then there is a sublattice $N_2 \subset N$ with $N = N_1 \oplus N_2$.*

This is Exercise 1.3.5 in [1].

Proof The classification theorem for free modules over PID's allows to deduce that ${}^{N}/_{N_1}$ is free. Let $B \subset N$ denote the elements that map to a basis of ${}^{N}/_{N_1}$ under the projection. We set $N_2 = \{\mathbb{Z}\text{-linear combinations in } B\}$. We conclude by noting that $N_1 \times N_2 \to N$,$(n_1, n_2) \mapsto n_1 + n_2$ is bijective. It has trivial kernel since $n_1 + n_2 = 0$ implies $n_2 \in N_1$ so by arguing via the basis, $n_2 = 0$, so n_1 is also 0. If $y \in N$, then again arguing via B, there in $n_2 \in N_2$ with $y - n_2 \in N_1$, from which bijectivity follows. $\square$

Bibliography

[1] Cox David, Little John, Schenk Hal . *Toric Varieties*. American Mathematical Society, 2010.

[2] Geert Popma. Toric geometry, 2014-2015.

[3] Joseph J. Rotman. *Advanced Modern Algebra*. Pearson Education, 2002.

[4] Philip Sandt. Removal of singularities in toric varieties. 2023.