Chow groups

Definition of Chow groups

Let X be a variety.

Define $Z_k(X)$ $Z_k(X)\{V \subset X : V \text{ is a } k\text{-dimensional subvariety}\}$.

We can define an equivalence relation on $Z_k(X)$ as follows:
$V_1 \sim V_2$ if there exists $W \subset X$ of dimension $k+1$ and $f \in K^*(W)$ such that $V_1 - V_2 = \mathrm{div} f$

With this equivalence relation we can define the Chow groups:

Definition - For a variety X, its Chow groups are $A_k(X) = Z_k(X)/\sim$

Notice that in general, $A_k(X)$ is very difficult to understand

Basic Structures

Let X be an n-dimensional variety.

Fundamental class

To compute $A_n(X)$, notice that there is no $W \subset X$ of dimension $n+1$, hence $A_n(X) = Z_n(X)$.
This is the free abelian group generated by the irreducible components of X. But X, is an n-dimensional variety, so $[X]$ is the generator, ie. $1 = [X]$ is the fundamental class.

→ $A_n(X) = \mathbb{Z}$

Proper push-forward

Let $f : X \to Y$ be a proper map

Then there is an induced map

$$f_* : A_k(X) \to A_k(Y)$$
$$[Z] \mapsto [f(Z)]$$

which is called the proper push-forward.

More precisely,

$$[Z] \mapsto \begin{cases} \deg(V/f(V)) \cdot [f(Z)] & \text{if } \dim f(Z) = \dim Z \\ 0 & \text{otherwise} \end{cases}$$

but for our purposes we can assume the induced map looks like the first map.

Excision

Let $U \subset X$ be open, let $Z = X - U$

Then we have an exact sequence

$$A_k(Z) \to^{j_*} A_k(X) \to^{i^*} A_k(U) \to 0$$

Notice that for divisors, we had the exact sequence $A_{n-1}(Z) \to A_{n-1}(X) \to A_{n-1}(U) \to 0$, which exactly what we get here.

Main Theorem

Remember that on an arbitrary toric variety X, the Weil divisors generate the group $A_{n-1}(X) = \mathrm{WDiv}/K^*(X)$.
We want to generalise this to Chow groups A_k for $k \neq n-1$:

Theorem - The Chow group $A_k(X)$ of an arbitrary toric variety $X = X(\triangle)$ is generated by the classes of the orbit closures $V(\sigma)$ of the cones σ of dimension $n-k$ of $\triangle$.

Proof

- Define $X_i \subset X$ as $X_i = \cup_{\sigma:\dim(\sigma)\geq n-i} V(\sigma)$
  ie the union of the closed subvarieties corresponding to cones of dimension at least $n-i$.
- This gives the following chain of inclusions of subvarieties:
  $X = X_n \supset X_{n-1} \supset \cdots \supset X_{-1} = \emptyset$
- With theses definitions we have $X_i - X_{i-1} = \cup_{\sigma:\dim(\sigma)=n-i} \mathcal{O}_\sigma$
  ie the disjoint union of orbits of cones of dimension $n-i$
- Using the excision property we had above, this gives the exact sequence

$$A_k(X_{i-1}) \to A_k(X_i) \to \oplus_{\dim \sigma = n-i} A_k(\mathcal{O}_\sigma) \to 0$$

- Now we have that for an n-torus T, $A_k(T) = 0$ for $k \neq n$ and $A_n(T) = \mathbb{Z} \cdot [T]$
  Using that $\mathcal{O}_\sigma$ are tori combined with what we saw above, we get for $k = i$

$$A_k(X_{i-1}) \to A_k(X_i) \to \oplus_{\dim \sigma = n-i} \mathbb{Z} \cdot [\mathcal{O}_\sigma] \to 0$$

- Thus we can inductively show that
  - $A_k(X_i) = 0$ if $k > i$
  - $A_k(X_i) \cong \oplus_{\dim \sigma = n-i} \mathbb{Z} \cdot [\mathcal{O}_\sigma]$ for $k \leq i$
- Notice that the restriction from $A_k(X_i)$ to $A_k(\mathcal{O}_\sigma)$ maps $[V(\sigma)] \mapsto [\mathcal{O}_\sigma]$,
- Therefore the classes $[V(\sigma)]$ for $\dim(\sigma) = n - k$ generate $A_k(X_i)$ for $k \leq i$, hence generate $A_k(X)$

Definition of Chow rings

From now on suppose X is non-singular.

Definition - The Chow ring for non-singular varieties is defined as follows:
$A^k(X) := A_{n-k}(X)$.

Theorem - $A^*(X) = \oplus A^k(X)$ is a ring.

These Chow rings have a ring structure attached to them:
Let $x \in A^k(X)$ and $y \in A^j(X)$, then $xy \in A^{k+j}(X)$.

Definition - If Z, W are subspaces of X, then they intersect transversely if $T_x Z \oplus T_x W = T_x X$

On a toric variety, $V(\sigma)$ and $V(\tau)$ intersect transversely, unless $V(\tau) \subset V(\sigma)$

If $x = [Z]$, $y = [W]$ and Z, W are transverse, then $xy = [Z \cap W]$.
If Z, W are not transverse, then we have to shift them slightly to get $Z' \sim Z$ and $W' \sim W$ such that they are transverse and hence $[Z] \cdot [W] = [Z' \cap W']$

We now have a Chow group A_* which is a module over the Chow ring A^*.

Analogy to Topology

$$A_* \longleftrightarrow H_*$$

ie the chow groups correspond to homology groups.

$$A^* \longleftrightarrow H^*$$

ie the Chow rings correspond to cohomology rings.

$$H_* \text{ module over } H^* \longleftrightarrow A_* \text{ module over } A^*$$