Divisors

Definitions

Definition - On any variety X, a **Weil divisor** is a finite formal sum $\sum a_i V_i$ of irreducible, closed subvarieties V_i of codimension one in X.

Definition - A **Cartier divisor** $D = (U_\alpha, f_\alpha)$ is given by the data of a covering of $X = \cup U_\alpha$ by affine open sets U_α, and nonzero rational functions f_α on U_α called **local equation**, such that the ratios f_α / f_β are nowhere zero regular (holomorphic) functions on $U_\alpha \cap U_\beta$.

A nonzero rational function f determines a **principal divisor** $\mathrm{div}(f)$ whose local equation in each open set is f.
The group of principal divisors is a subgroup of the group of Cartier divisors.

Relationship

A Cartier divisor D determines a Weil divisor, denoted $[D]$ by

$$[D] = \sum_{\mathrm{cod}(V,X)=1} \mathrm{ord}_V(f_\alpha) \cdot V$$

where $\mathrm{ord}_V(f_\alpha)$ is the order of vanishing of an equation f_α for D in the local ring along the subvariety V.

Divisors on toric varieties

We are interested in divisors on toric varieties $X(\triangle, N)$ that are mapped to themselves by the torus.

We want to show the following:

$$\begin{aligned} T\text{-Weil divisors } &\simeq \oplus_{\tau \in \triangle(1)} \mathbb{Z} \\ T\text{-Cartier divisors} &\simeq PL(\triangle) \end{aligned}$$

T-Weil divisors

They are of the form

$$\sum a_i[D_i]$$

such that the $[D_i]$ are irreducible subvarieties of codimension one in X that are T-stable, ie fixed by T.

By the Orbit-cone correspondence, orbits of codimension one under the action of T correspond to cones of dimension one.

As the cones of dimension one are the rays of the fan, for each ray $\tau_i \in \triangle(1)$, the corresponding orbit closure $V(\tau_i)$ (a union of torus orbits) is an irreducible subvariety of codimension one, so $D_i = V(\tau_i)$.

Thus the T-Weil divisors are exactly the linear combinations of these D_i defined by orbit closures.

→ $T\text{-Weil divisors} \simeq \oplus_{\tau_i \in \triangle(1)} \mathbb{Z}[D_i]$

T-Cartier=PL

Grading = Torus action

We will assume the following, which is left as an exercise:

Grading by $M \iff$ Torus action by $\mathrm{Spec}\mathbb{C}[M]$

A Cartier divisor D on $U_\sigma = \mathrm{Spec}\mathbb{C}[S_\sigma]$ is given by an $I \subset Q(\mathbb{C}[S_\sigma])$, by definition of Cartier divisors.

The divisor D is invariant under the torus action.
Thus, using the property above, it is graded by M, ie a direct sum of spaces $\mathbb{C} \cdot \chi^u$ over some set of $u \in M$.

$$I = \oplus_{u \in M} I_u$$

$Q(\mathbb{C}[S_\sigma])$ has a grading, it's graded pieces are 1-dimensional, hence the I_u must be one-dimensional.

We know that $\mathbb{C}[M] = \oplus_{u \in M} \mathbb{C}\chi^u$

Hence the only chance is that I_u is $\mathbb{C}\chi^u$ or 0.

From the Orbit-Cone correspondence we had that there is a distinguished point x_σ where I is principal, so we must have that I/mI is 1-dimensional, where m is the sum of all $\mathbb{C} \cdot \chi^u$ for $u \neq 0$.

So we have proven that I is determined by a unique $u \in M$

Thus a cartier divisor on U_σ has the form $\operatorname{div}(\chi^u)$ for some unique $u \in M$.

Map T-Cartier $\to$ T-Weil

For affine toric varieties U_σ, we can define a map from T-Cartier to T-Weil divisors:

$$\operatorname{div}(\chi^u) \mapsto \sum_{v_i} \langle u, v_i \rangle [D_{\tau_i}]$$

where v_i is the primitive vector on τ_i.

Proof -

- the order can be calculated on the open set $U_\tau \cong \mathbb{C} \times (\mathbb{C}^*)^{n-1}$, on which $V(\tau)$ corresponds to $(0) \times (\mathbb{C}^*)^{n-1}$
- this reduces the calculations to the one-dimensional case, ie where $N = \mathbb{Z}$, τ is generated by $v = 1$, and $u \in M = \mathbb{Z}$.
- then χ^u is the monomial X^u, whose order of vanishing at the origin is u.

T-Cartier on U_σ

Let $M(\sigma) = \sigma^\perp \cap M$.

We want to show that T-Cartier divisors on U_σ correspond to elements of $M/M(\sigma)$.

If σ is full-dimensional, $\sigma^\perp = \{0\}$, so $M/\sigma^\perp \cap M = M$, and we saw that a T-Cartier divisor on U_σ corresponds to an element of M.

If σ is not full-dimensional, then

$$\begin{aligned} &\operatorname{div}(\chi^u) = \operatorname{div}(\chi^{u'}) \\ \iff &\langle u - u', v_i \rangle = 0 \ \ \forall \tau_i \\ \iff &\langle u - u', w \rangle = 0 \ \ \forall w \in \sigma \\ \iff &u - u' \in \sigma^{\perp} \cap M = M(\sigma) \end{aligned}$$

Therefore T-Cartier divisors on U_σ correspond to elements of $M/M(\sigma)$.

$\{T\text{-Cartier divisors on } U_\sigma\} \cong \ M/M(\sigma)$

On a general toric variety $X(\triangle, N)$ we can thus define T-Cartier divisors by specifying an element $u(\sigma) \in M/M(\sigma)$ for each cone $\sigma \in \triangle$, which then define divisors $\operatorname{div}(\chi^{u(\sigma)})$ on U_σ.
The divisors must agree on overlaps, ie when τ is a face of σ, $u(\sigma)$ must map to $u(\tau)$ under the canonical map $M/M(\sigma) \to M/M(\tau)$.

T-Cartier divisors $\simeq PL(\triangle)$

Consider a real-valued function $h : |\triangle| \to \mathbb{R}$ on the support $|\triangle| = \cup_{\sigma\in\triangle}\sigma$. If it is a piecewise linear function $\triangle$, then it is

- linear on each $\sigma \in \triangle$
  this means there exists $l_\sigma \in M$ for each $\sigma \in \triangle$ such that $h(n) = \langle l_\sigma, n \rangle$ for $n \in \sigma$
- $\mathbb{Z}$-valued on $N \cap |\triangle|$
- $\langle l_\sigma, n \rangle = \langle l_\tau, n \rangle$ holds whenever $n \in \tau < \sigma$

We saw earlier that $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \subset \operatorname{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R}) = M_{\mathbb{R}}$.
Combining this with the fact that Cartier divisors are specified by $u(\sigma) \in M/M(\sigma)$ and that they agree on overlaps, we get the desired correspondence:

$$T\text{-Cartier divisors} \simeq PL(\triangle)$$

Using this we can define the following map:

$$PL(\triangle) \to \oplus_{\tau \in \triangle(1)} \mathbb{Z}\tau$$
$$f \mapsto \sum_{\tau_i} f(v_i) \cdot \tau_i$$

Special cases

$\triangle$ non-singular / smooth varieties

Consider the map

$$PL(\triangle) \to \oplus_{\tau_i \in \triangle(1)} \mathbb{Z}$$
$$h \mapsto h(v_i)$$

This is an injective homomorphism.

If σ is non-singular, the v_i are part of a $\mathbb{Z}$-basis for N, so there exists some $u \in M$ such that $\langle u, v_i \rangle = h(v_i)$, hence the homomorphism is surjective.

Therefore, for smooth varieties,

$$\text{T-Weil divisors} \simeq \text{T-Cartier divisors}$$