PICARD GROUPS OF TORIC VARIETIES

1. General Algebraic Geometry

We will work with an n-dimensional variety X. We have seen in the previous lecture the notion of Weil and Cartier divisors. These are written as $\mathrm{WDiv}(X)$ and $\mathrm{CDiv}(X)$.

Eva explained that every (non-zero) rational function $f \in K^*(X)$ gives a divisor, by setting

$$\mathrm{div}(f) = \sum_Z \mathrm{ord}_V(f)[Z]$$

Definition 1. The *Chow group* of dimension $n-1$ cycle is the group

$$A_{n-1}(X) = \mathrm{WDiv}(X)/K^*(X)$$

Definition 2. The Picard group of X is the quotient of Cartier divisors modulo principal ones,

$$\mathrm{Pic}(X) = \mathrm{CDiv}(X)/K^*(X)$$

Thus, two Weil or Cartier divisors become equivalent in A_{n-1} or Pic repsectively if they differ by the divisor of a rational function. Divisors of rational functions are sometimes called principal.

The group $A_{n-1}(X)$ has a very important property, called excision.

Theorem 1.1. *Let $Z \subset X$ be closed, and $U = X - Z$ its open complement. Then we have an exact sequence*

$$A_{n-1}(Z) \longrightarrow A_{n-1}(X) \longrightarrow A_{n-1}(U) \longrightarrow 0$$

Before giving the proof, let's discuss the maps. The map $A_{n-1}(X) \to A_{n-1}(U)$ sends $\sum a_i Z_i$ to the restriction $\sum a_i Z_i|_U$. This is well-defined, because if $\sum a_i Z_i = \mathrm{WDiv}(f)$, then so is $\sum a_i Z_i|_U$: the fraction field of X and U is the same. The map $A_{n-1}(Z)$ takes a class $\sum a_i W_i$ and simply includes it into X.

Proof. The fact is deep, but the proof is easy. Suppose

$$\sum a_i[Z_i]$$

is a divisors on U. Then we can get a divisor on X simply by closure,

$$\sum a_i[\overline{Z_i}]$$

So the map $A_{n-1}(X) \to A_{n-1}(U)$ is surjective. Let us next suppose that

$$\sum a_i[Z_i]|_U = \mathrm{div}(f)$$

Then, since the fraction field of X and U is the same

$$\sum a_i[Z_i] - \mathrm{div}(f)$$

is the 0 divisor on U, hence is supported on Z. This proves the claim. ◊

2. Toric Varieties

The excision sequence has the following very nice consequence for toric varieties. Let us assume $X = X(\Delta, N)$ is a toric variety, $U = T$ is the torus, and $Z = X - T$ is the boundary. Then we apply our proposition and find

$$A_{n-1}(Z) \longrightarrow A_{n-1}(X) \longrightarrow A_{n-1}(T) \longrightarrow 0$$

This is good news, because $T = \operatorname{Spec} \mathbb{C}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$, which is the spectrum of a unique factorization domain, and we have

Lemma 2.1. *Suppose A is a UFD, $U = \operatorname{Spec} A$. Then*

$$A_{n-1}(U) = 0$$

Proof. A divisor on U corresponds to a sum of codimension 1 subvarieties. These correspond to prime ideals of height 1, i.e. p such that there is no prime q with $0 \subset q \subset p$. We claim that all such p are principal: This is because an ideal is prime if and only if it is generated by a collection of relatively prime and irreducible elements $(f_1, \cdots, f_m)$, and such ideals have height 1 if and only if they are generated by exactly one element. $\Diamond$

This means that our sequence above simplifies, and we have

$$A_{n-1}(Z) \to A_{n-1}(X) \to 0$$

But $A_{n-1}(Z)$ is nothing but the T-Weil divisors from Eva's talk: it is

$$A_{n-1}(Z) = \mathrm{WDiv}^T(X)$$

and thus $A_{n-1}(X)$ is the quotient of that by the T-Weil divisors which are principal. Eva also showed that these are of the form

$$\mathrm{WDiv}(\chi^u)$$

for $u \in M$. So we have found

$$A_{n-1}(X) = \mathrm{WDiv}^T(X)/M = \oplus_{\rho \in \Delta(1)} \mathbb{Z}\rho/M$$

Corollary 2.2. *For a toric variety $X = X(\Delta, N)$, we have*

$$\mathrm{Pic}(X) = \mathrm{CDiv}^T(X)/M$$

Proof. We have a diagram

$$\begin{array}{ccc} \mathrm{CDiv}^T & \longrightarrow & \mathrm{CDiv} \\ \downarrow & & \downarrow \\ \mathrm{WDiv}^T & \longrightarrow & \mathrm{WDiv} \end{array}$$

The diagram is a fiber product: The elements in $\mathrm{CDiv}^T(X)$ are the elements in CDiv that map into the subgroup WDiv^T of WDiv. Consider an element L of $\mathrm{Pic}(X)$. It is an equivalence class of a Cartier divisor $[D]$, up to rational equivalence. Since $A_{n-1}(X)$ is generated by T-invariant Weil-divisors, L is in fact equivalent to a T-invariant Cartier divisor $[E]$, up to rational equivalence. Since rational equivalence is the same for Weil and Cartier divisors, we can conclude that $\mathrm{CDiv}^T(X)$ surjects onto $\mathrm{Pic}(X)$, and the kernel consists of T-invariant rational functions, i.e. M. $\Diamond$

Thus, we get

Theorem 2.3 (Main Theorem)**.** *Let* $X = X(\Delta, N)$ *be a toric variety.*

$$\mathrm{Pic}(X) = \mathrm{PL}(\Delta)/M$$

Proof. From Eva's talk we know

$$\mathrm{CDiv}^T = \mathrm{PL}(\Delta)$$

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From here on, do several examples.