Exercise Sheet 10

1. Consider $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ with respect to the Lebesgue measure. Show that the canonical map $L^1(\mathbb{R}) \to L^\infty(\mathbb{R})^*$ is not surjective.

   Hint: Use the Hahn-Banach theorem to extend the functional

   $$f \in C^b(\mathbb{R}) \mapsto f(0) \in \mathbb{R}.$$

2. Let E be a normed space that is separable. Show that $B^{E^*}_{\leqslant 1}(0)$ is metrizable in the weak*-topology.

3. Consider the space

   $$c_0(\mathbb{N}) := \{f\colon \mathbb{N} \to \mathbb{R} | \lim_{n\to\infty} = 0\}$$

   with the sup-norm. Identify $c_0(\mathbb{N})^*$.

4. Show that the canonical map $\ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})^*$ is not surjective.

   Hint: Consider the sequence

   $$\lambda_n\colon \ell^\infty(\mathbb{N}) \to \mathbb{R}$$
   $$\lambda_n(g) = \frac{1}{n}\sum_{i=1}^{n} g(k).$$

5. Let E be a complex vector space. Formulate conditions on a subset $A \subset E$ so that the construction in Prop. VI. 6 leads to a seminorm on E.