Exercise Sheet 10 - Solutions

1. Consider $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ with respect to the Lebesgue measure. Show that the canonical map $L^1(\mathbb{R}) \to L^\infty(\mathbb{R})^*$ is not surjective.

   Hint: Use the Hahn-Banach theorem to extend the functional

   $$f \in C^b(\mathbb{R}) \mapsto f(0) \in \mathbb{R}.$$

   Solution: Let $\phi\colon C^b(\mathbb{R}) \to \mathbb{R}$ be the functional $\phi(f) = f(0)$. Note that $\phi(f) \leqslant ||f||_\infty$, so the Hahn-Banach theorem implies that there is $\Phi\colon L^\infty(\mathbb{R}) \to \mathbb{R}$ with $||\Phi(f)|| \leqslant ||f||_\infty$ for all $f \in L^\infty(\mathbb{R})$. This implies that Φ is continuous.

   Suppose there is $f \in L^1(\mathbb{R})$ with $\Phi(g) = \int_\mathbb{R} f(x)g(x)dx$. Let $K \subset \mathbb{R}$ be a compact subset such that $0 \notin K$. There exists a sequence of bump functions $(\varphi_n) \in C_c(\mathbb{R})$ such that $\varphi_m \xrightarrow{L^\infty} 1_K$ and $\varphi_n(0) = 0$ for all $n \in \mathbb{N}$. We obtain

   $$\Phi(1_K) = \lim_{n\to\infty} \Phi(\varphi_n) = \lim_{n\to\infty} \phi(\varphi_n) = \lim_{n\to\infty} \varphi_n(0) = 0.$$

   In other words, this means

   $$\int_K f(x)dx = 0$$

   for each compact subset $K \subset \mathbb{R}$ with $0 \notin K$. Because the set $\{0\}$ is a null set, the above equality holds for each compact subset K. The Lebesgue measure is regular, so this implies

   $$\int_A f(x)dx = 0$$

   for each measurable $A \subset \mathbb{R}$. Define the sets

   $$A_\epsilon = \{x \in \mathbb{R} | f(x) \geqslant \epsilon\}$$
   $$B_\epsilon = \{x \in \mathbb{R} | f(x) \leqslant -\epsilon\}$$

   for all $\epsilon > 0$. By the above equality

   $$0 = \int_{A_\epsilon} f(x)dx \geqslant \epsilon\mu(A_\epsilon)$$

   thus $\mu(A_\epsilon) = 0$ for each $\epsilon > 0$. Similarly, we get $\mu(B_\epsilon) = 0$ for all $\epsilon > 0$. But this implies $f = 0$ (up to a null set) and hence $\Phi = 0$. This is a contradiction to, say, $\Phi(1) = 1$.

2. Let E be a normed space that is separable. Show that $B^{E^*}_{\leqslant 1}(0)$ is metrizable in the weak*-topology.

   Solution: Let $\{v_n : n \in \mathbb{N}\} \subset E$ be a dense countable subset and define $X := \prod_{n\in\mathbb{N}} D_{v_n}$. Consider the map

   $$B^{E^*}_{\leqslant 1}(0) \to X$$

from the proof of Theorem V. 29. This map is injective because the $\{v_n\}$ lie densely in E. The proof of Theorem V. 29. shows that this map is open onto its image, so it defines a homeomorphism of the image with $B^{E^*}_{\leqslant 1}(0)$ equipped with the weak*-topology. If we prove that X is metrizable, then the ball is metrizable because the weak*-topology on $B^{E^*}_{\leqslant 1}(0)$ will be induced by the restriction of the metric on X.

We construct a metric on the product space X with the formula

$$d((s_v),(t_v)) := \sum_{n=1}^{\infty} \frac{|s_{v_n} - t_{v_n}|}{||v_n||2^n}.$$

By the dominated convergence theorem, the function $d\colon X \times X \to \mathbb{R}$ is continuous. This implies that the product topology on X is finer than the topology induced by d. Let $n \in \mathbb{N}$, $t \in D_{v_n}$ and $\epsilon > 0$. Set

$$U := D_{v_1} \times \cdots \times D_{v_{n-1}} \times B_\epsilon(t) \times D_{v_{n+1}} \times \cdots \subset X.$$

Let $x \in U$ and define

$$\tilde{x} := (x_0, \ldots, x_{n-1}, t, x_{n+1}, \ldots).$$

For each $y \in X$ with $d(\tilde{x}, y) < \frac{\epsilon}{||v_n||2^n}$ we have

$$|t - y_{v_n}| < \epsilon.$$

This implies

$$B^d_{\frac{\epsilon}{||v_n||2^n}}(\tilde{x}) \subset U.$$

Because $x \in U$, we have $x \in B^d_{\frac{\epsilon}{||v_n||2^n}}(\tilde{x})$. Now $x \in U$ was arbitrary which means we can write U as a union of open balls in the metric topology. In particular, U is open in the metric topology. Since such opens U generate the product topology, this means the metric topology is finer than the product topology. On a gagné!

3. Consider the space

$$c_0(\mathbb{N}) := \{f\colon \mathbb{N} \to \mathbb{R} | \lim_{n\to\infty} = 0\}$$

with the sup-norm. Identify $c_0(\mathbb{N})^*$.

Solution: We have a continuous map

$$J\colon \ell^1(\mathbb{N}) \to c_0(\mathbb{N})^*$$
$$J(\phi)(f) := \sum_{n\geqslant 0} \phi(n) f(n).$$

Note that it is injective. If we prove that it is surjective, then the open mapping theorem implies that it is an isomorphism of Banach spaces.

Let $\Phi \in c_0(\mathbb{N})^*$ and define the function $\phi(n) := \Phi(1_{\{n\}})$ for all $n \in \mathbb{N}$. For each $f \in c_0(\mathbb{N})$ and $N \in \mathbb{N}$, we define the functions $f_N := f|_{\{0,\ldots,N\}}$. Because $\lim_{n\to\infty} f(n) = 0$, we get $f_N \to f$ as $N \to \infty$. This implies

$$\Phi(f) = \lim_{N\to\infty} \Phi(f_N) = \lim_{N\to\infty} \Phi\left(\sum_{n=1}^{N} f(n)1_{\{n\}}\right) = \sum_{n=1}^{\infty} f(n)\phi(n).$$

For $N \geqslant 0$, we define the function

$$g_N(n) := \begin{cases} \operatorname{sgn}(\phi(n)) & \text{if } n \leqslant N \\ 0 & \text{else} \end{cases}$$

By the equality above, we get

$$\Phi(g_N) = \Phi\left(\sum_{i=0}^{N} g_N(i)1_{\{i\}}\right) = \sum_{n=0}^{N} |\phi(n)|.$$

We have $|\Phi(g_N)| \leqslant ||\Phi||$, so the monotone convergence theorem implies

$$\sum_{n=0}^{\infty} |\phi(n)| < \infty.$$

This implies $\phi \in \ell^1(\mathbb{N})$ hence the above equality gives

$$\Phi = J(\phi).$$

4. Show that the canonical map $\ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})^*$ is not surjective.

   Hint: Consider the sequence

$$\lambda_n \colon \ell^\infty(\mathbb{N}) \to \mathbb{R}$$
$$\lambda_n(g) = \frac{1}{n}\sum_{i=1}^{n} g(k).$$

Solution: Let

$$V := \{f \in \ell^\infty(\mathbb{N}) | \lim_{n\to\infty} \lambda_n(f) \text{ exists}\}.$$

This is a subspace $V \subset \ell^\infty(\mathbb{N})$ which comes with the functional

$$\lambda(f) := \lim_{n\to\infty} \lambda_n(f).$$

By the Hahn-Banach theorem, there exists a functional $\Lambda \colon \ell^\infty(\mathbb{N}) \to \mathbb{R}$ which extends λ and satisfies $|\Lambda(f)| \leqslant ||f||$ for each $f \in \ell^\infty(\mathbb{N})$. This means Λ is continuous.

Suppose Λ lies in the image of the canonical map $\ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})^*$. Then there exists $F \in \ell^1(\mathbb{N})$ such that

$$\Lambda(f) = \sum_{n=1}^{\infty} F(n) f(n).$$

Note that the indicator function $1_{\{n\}} \in V$ for each $n \in \mathbb{N}$. We get

$$0 = \lambda(1_{\{n\}}) = \Lambda(1_{\{n\}}) = F(n).$$

This implies $\Lambda = 0$ which is a contradiction to $\Lambda(1) = \lambda(1) = 1$.

5. Let E be a complex vector space. Formulate conditions on a subset $A \subset E$ so that the construction in Prop. VI. 6 leads to a seminorm on E.

   Solution: Let $A \subset E$ be a subset. We impose the following conditions on A :

   (a) For all $t \in [0,1]$ and $v, w \in A$ we have $tv + (1-t)w \in A$.
   (b) $0 \in A$
   (c) For each $v \in E$ there exists $\alpha \in \mathbb{R}$ such that $v \in \lambda A$ for each $\lambda \in \mathbb{R}$ with $\lambda \geqslant \alpha$.
   (d) For all $u \in \mathbb{C}$ with $|u| = 1$ we have $uA = A$.

   These conditions imply the conditions of Prop. VI.6 (when one regards $\mathbb{R}$ as a real vector space), so the function

$$p(v) := \inf\{t > 0 | v \in tA\}$$

   is well-defined and gives a gauge function. We have to prove $p(\lambda v) = |\lambda| p(v)$ for all $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ and $v \in E$. Note that

$$\lambda v \in tA \Leftrightarrow v \in \lambda^{-1} tA \Leftrightarrow v \in |\lambda^{-1}| tA \Leftrightarrow |\lambda| v \in tA$$

   So we get

$$p(\lambda v) = p(|\lambda| v) = |\lambda| p(v).$$