Exercise Sheet 11 - Solutions

1. Let Γ be a group with a normal subgroup $N \lhd \Gamma$ such that N and Γ/N satisfy the conclusion of the Markov-Kakutani fixed point theorem. Show that it holds for Γ as well.

   Solution: Let V be a topological vector space generated by a sufficient family of seminorms with a continuous action by the discrete group Γ. Suppose $A \subset V$ is a convex, compact, non-empty and Γ-invariant subset. Define

   $$A^N := \{v \in A | gv = v \ \forall g \in N\}.$$

   The MK-fixed point theorem says $A^N \neq \emptyset$. Note that for each $v \in A^N$, $g \in \Gamma$ and $h \in N$ we have

   $$h(gv) = g((g^{-1}hg)v) = gv$$

   because N is normal in Γ. This says that A^N is Γ-invariant. By definition, the subgroup N acts trivially on A^N, so the Γ-action on A^N factors through Γ/N. Moreover, A^N is closed in A (because N acts by continuous automorphisms) and convex. So the MK-fixed point theorem says that there is $v \in A^N$ such that $gNv = g$ for all $gN \in \Gamma/N$. In particular, there is $v \in A$ such that $gv = v$ for all $g \in \Gamma$.

2. Show that if Γ has property (F) (Remark VI.24) then it satisfies the conclusions of the MK-fixed point theorem.

   Solution: Let $A \subset V$ be compact, convex, Γ-invariant, and non-empty. For each finite $F \subset \Gamma$ we define the map $T_F \colon A \to A$ by

   $$T_F(v) := \frac{1}{|F|} \sum_{g \in F} gv$$

   for all $v \in A$.

   By property (F), there exists a sequence $(F_n)_n$ of subset of Γ with

   $$\frac{|\gamma F_n \triangle F_n|}{|F_n|} \to 0$$

   as $n \to \infty$. Let $v_0 \in A$. There exists a subsequence of $(T_{F_n} v_0)_n$ which converges to some $v \in A$ because A is compact. Let $||\cdot||$ be a continuous seminorm on V. Because A is compact there exists $C \geqslant 0$ with $||w|| \leqslant C$ for all $w \in A$. For each $\gamma \in \Gamma$ we have

   $$\begin{aligned} ||\gamma v - v|| &= \left|\left| \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \gamma g v_0 - g v_0 \right|\right| \\ &\leqslant \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g \in \gamma F_n \triangle F_n} ||g v_0|| \\ &\leqslant \lim_{n\to\infty} \frac{C|\gamma F_n \triangle F_n|}{|F_n|} = 0. \end{aligned}$$

3. Determine the extreme points of $M^1([0,1])$.

   Solution: Let μ be an extreme point of $M^1([0,1])$. Suppose the support of μ is not a point. There exists a decomposition $[0,1]=[0,t)\sqcup[t,1]$ such that $\mu|_{[0,t)}\neq 0$ and $\mu|_{[t,1]}\neq 0$. We can write
$$\mu=\mu([0,t))\frac{\mu|_{[0,t)}}{\mu([0,t)}+\mu([t,1])\frac{\mu|_{[t,1]}}{\mu([t,1])}.$$
   This is a contradiction to μ being an extreme point. Therefore the support of μ is a point and μ is a Dirac measure.

   Let $x\in[0,1]$ and consider the Dirac measure δ_x supported at x. Suppose there exist $\mu_1,\mu_2\in M^1([0,1])$ and $t_1,t_2\in(0,1)$ with $t_1+t_2=1$ and
$$\delta_x=t_1\mu_1+t_2\mu_2.$$
   Let $U\subset[0,1]$ be an open subset such that $x\notin U$. Then
$$0=\delta_x(U)=t_1\mu_1(U)+t_2\mu_2(U).$$
   This implies $\mu_1(U)=\mu_2(U)=0$ because $t_i\neq 0$. This says μ_1 and μ_2 are supported at x and hence $\mu_1=\mu_2=\delta_x$.

4. Let $\varphi\colon[0,1]\to[0,1],\varphi(x):=x^2$; determine all the φ-invariant probability measures.

   Solution: Let μ be a φ-invariant probability measure. For each $t\in(0,1)$ define $I_t:=(0,t)$. Note that $\varphi(I_t)=I_{t^2}$, so we get
$$\mu(I_t)=\mu(I_{t^{2^n}})$$
   for each $n\geqslant 1$. The dominated convergence theorem implies that, as $n\to\infty$, this equality converges to
$$\mu(I_t)=0$$
   for each $t\in(0,1)$. Thus $\mu=t_0\delta_0+t_1\delta_1$ where δ_0 and δ_1 are the Dirac measures supported at 0 and 1 respectively and $t_0+t_1=1$.

5. In Example VI. 26 justify why for $\alpha\notin\mathbb{Q}/\mathbb{Z}$, λ is the unique T_α-invariant probability measure.

   Solution: Let μ be a T_α-invariant probability measure. Define
$$T_m(f)(x):=\frac{1}{m}\sum_{k=0}^{m-1}f(x+k\alpha)$$
   for each $m\geqslant 1$ and $f\in C(S^1)$. Then
$$\int f d\mu=\int T_m f d\mu$$
   for all $f\in C(S^1)$ and $m\geqslant 1$ because μ is T_α-invariant. Define $\varphi_n(x):=e^{2\pi inx}$ for each $n\in\mathbb{Z}$. Because $\alpha\notin\mathbb{Q}/\mathbb{Z}$, we have
$$T_m(\varphi_n)(x)=\frac{\varphi_n(x)(e^{2\pi in\alpha m}-1)}{m(e^{2\pi in\alpha}-1)}$$

for all $n \neq 0$. This implies $T_m(\varphi_n) \to 0$ as $m \to \infty$ and, therefore,

$$\int_{S^1} \varphi_n d\mu = 0$$

for $n \neq 0$. Let $f \in C^\infty(S^1)$ then we can write for each $x \in S^1$

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\varphi_n(x).$$

Because f is smooth the Fourier series converges pointwise. Because S^1 is compact we get that the Fourier series converges uniformly. Thus dominated convergence implies

$$\int_{S^1} f d\mu = \int_{S^1} \sum_{n\in\mathbb{Z}} \hat{f}(n)\varphi_n(x)d\mu(x) = \sum_{n\in\mathbb{Z}} \int_{S^1} \hat{f}(n)\varphi_n(x)d\mu(x) = \hat{f}(0)\int_{S^1} d\mu = \hat{f}(0).$$

The smooth functions are dense in the space of continuous functions (convolve with a smooth bump function), so this determines $\mu = \lambda$.

6. Let $T \in SL_2(\mathbb{R})$ with real eigenvalues $\{\lambda, \lambda^{-1}\}$ with $\lambda > 1$. Then T induces a homeomorphism of $\mathbb{P}^1(\mathbb{R})$ still denoted T. Classify all T-invariant probablity measures on $\mathbb{P}^1(\mathbb{R})$. Prove that there exists no probability measure on $\mathbb{P}^1(\mathbb{R})$ which is invariant under $\mathrm{SL}_2(\mathbb{Z})$.

   Solution: There is $Q \in SL_2(\mathbb{R})$ with $QTQ^{-1} = \mathrm{diag}(\lambda^{-1}, \lambda)$. We begin by determining the $S := \mathrm{diag}(\lambda, \lambda^{-1})$-invariant probability measures. We denote by $\mathbb{P}^1(\mathbb{R})$ the set of lines in $\mathbb{R}^2$. This space is diffeomorphic to the quotient

   $$(\mathbb{R}^2\backslash\{0\})/\mathbb{R}^* \xrightarrow{\sim} \mathbb{P}^1(\mathbb{R})$$

   via the map which sends v to $v\mathbb{R}$ (by definition). This diffemorphism provides the homogeneous coordinates for $\mathbb{P}^1(\mathbb{R})$ where $[x : y] \in \mathbb{P}^1(\mathbb{R})$ denotes the line $(x, y)\mathbb{R}$ i.e. the line passing through (x, y). This space is compact because the inclusion $S^1 \to \mathbb{R}^2\backslash\{0\}$ induces a continuous surjection

   $$S^1 \twoheadrightarrow \mathbb{R}^2\backslash\{0\}.$$

   Let μ be a S-invariant probability measure. For each $0 \leqslant t < \infty$ we define the two subset

   $$I_t^+ := \{[1 : s] | s \in [0, t)\}$$
   $$I_t^- := \{[-1 : s] | s \in [0, t)\}.$$

   We have $S(I_t^\pm) = I_{\lambda^2 t}^\pm$ for all $0 \leqslant t \leqslant \infty$. This implies

   $$\mu(I_t^\pm) = \mu(I_0^\pm)$$

   for all $0 \leqslant t \leqslant \infty$. In particular, the measure μ is supported at the two points $[1, 0]$ and $[0, 1]$. These two points are fixed by S, so all the S-invariant probability measures are probability measures supported at these two points.

   Let μ be a T-invariant probability measure. The pullback $Q^*\mu$ is S-invariant, so it is supported at the two fixed points of S. This means μ is supported at the two fixed points of T (the eigenvectors). Any probability measure supported at these two points is automatically T-invariant, so these are all the T-invariant probability measures.