Exercise Sheet 12 - Solutions

1. Let $G \times X \to X$ be a countable group G acting on a compact Hausdorff space X by homeomorphisms. Show that the set $M^1(X)^G$ of G-invariant probability measures on X is a weak*-closed, convex subset of $M^1(X)$.

   Solution: Let $\mu_1, \mu_2 \in M^1(X)^G$ and $t_1 + t_2 = 1$. Then

   $$t_1\mu_1(X) + t_2\mu_2(X) = t_1 + t_2 = 1,$$

   so $t_1\mu_1 + t_2\mu_2$ is a probability measure.

   Let $(\mu_n)_n \in M^1(X)^G$ be a sequence converging to $\mu \in M^1(X)$ in the weak*-topology. In other words, for each $f \in C(X)$ we have

   $$\lim_{n\to\infty} \int_X f(x)d\mu_n(x) = \int_X f(x)d\mu(x).$$

   Let $g \in G$, then for each $f \in C(X)$

   $$\begin{aligned}\int_X f(gx)d\mu(x) &= \lim_{n\to\infty} \int_X f(gx)d\mu_n(x) \\ &= \lim_{n\to\infty} \int_X f(x)d\mu_n(x) \\ &= \int_X f(x)d\mu(x).\end{aligned}$$

2. Let (X, d) be a compact metric space and $(f_n)_n$ be a sequence of continuous functions converging pointwise to a continuous function $g\colon X \to \mathbb{R}$. Assume

   $$f_n(x) \leqslant f_{n+1}(x) \leqslant g(x)$$

   for all $n \geqslant 1$ and $x \in X$. Prove that $(f_n)_n$ converges uniformly to g.

   Solution: Define

   $$e_n := \max_{x\in X} \big(g(x) - f_n(x)\big).$$

   If we can prove $\lim_{n\to\infty} e_n = 0$ then f_n converges uniformly to g. Suppose

   $$\lim_{n\to\infty} e_n = e > 0.$$

   There exists a sequence $(x_n)_n \in X$ such that $g(x) - f_n(x) \geqslant e$ for each $n \geqslant 1$. Because X is compact, there exists a subsequence $(x_{n_k})_k$ which converges to some $x_0 \in X$. The pointwise convergence at x_0 implies that there is an open neighbourhood $x_0 \in U$ and $N \geqslant 1$ with

   $$g(x) - f_n(x) < e_0$$

   for all $n \geqslant N$ and $x \in U_0$. But this contradicts $x_{n_k} \in U$ for large enough k.

3. Prove Lemma VII.6.

Solution: Let $f \in L^1(\mathbb{R}^n)$ and $\xi, y \in \mathbb{R}^n$. Integration by substitution implies

$$\begin{aligned}(\lambda(y)f)^\wedge(\xi) &= \int_{\mathbb{R}^n} (\lambda(y)f)(x)e^{-i\langle x,\xi\rangle}dx \\ &= \int_{\mathbb{R}^n} f(x-y)e^{-i\langle x,\xi\rangle}dx \\ &= \int_{\mathbb{R}^n} f(x)e^{-i\langle x+y,\xi\rangle}dx = e^{-i\langle y,\xi\rangle}\widehat{f}(\xi).\end{aligned}$$

Using the same notation as before we get

$$\begin{aligned}(\lambda(y)\widehat{f})(\xi) &= \widehat{f}(\xi+y) \\ &= \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi+y\rangle}dx \\ &= \int_{\mathbb{R}^n} (e^{-i\langle x,y\rangle}f(x))e^{-i\langle x,\xi\rangle}dx = (e^{-i\langle -,y\rangle}f)^\wedge(\xi)\end{aligned}$$

Let $a \neq 0$ then integration by substitution implies

$$\begin{aligned}(\delta_a f)(\xi) &= \int_{\mathbb{R}^n} (\delta_a f)(x)e^{-i\langle x,\xi\rangle}dx \\ &= \int_{\mathbb{R}^n} f(x/a)e^{-i\langle x+y,\xi\rangle}dx \\ &= |a|^n \int_{\mathbb{R}^n} f(x)e^{-i\langle ax,\xi\rangle}dx = |a|^n\delta_{1/a}\widehat{f}(\xi).\end{aligned}$$

By integration by substitution, we get

$$\begin{aligned}(\delta_a \widehat{f}) &= \widehat{f}(\xi/a) \\ &= \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi/a\rangle}dx \\ &= |a|^n \int_{\mathbb{R}^n} f(ax)e^{-i\langle x,\xi\rangle}dx = |a|^n(\delta_{1/a}f)^\wedge(\xi).\end{aligned}$$