

Exercise Sheet 13

1. Prove Lemma VII.13.
2. Prove Lemma VII.17.
3. Prove the following formulae for all $f_1, f_2, f_3 \in L^1(\mathbb{R}^n)$

$$\begin{aligned}f_1 * f_2 &= f_2 * f_1, \\(f_1 * f_2) * f_3 &= f_1 * (f_2 * f_3).\end{aligned}$$

4. Let $\|\cdot\|_{2,k}$ be a Sobolev norm on $W^{2,k}(\mathbb{R}^n)$. Define

$$\|f\| := \|(1 + \|\xi\|^k)\hat{f}\|_{L^2}$$

for all $f \in W^{2,k}(\mathbb{R}^n)$. Show that the norms $\|\cdot\|$ and $\|\cdot\|_{2,k}$ are equivalent on $W^{2,k}(\mathbb{R}^n)$.

5. Let

$$\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

be the Laplacian on \mathbb{R}^n . Show that $\Delta - \text{id}: C_{00}(\mathbb{R}^n) \rightarrow C_{00}(\mathbb{R}^n)$ extends to an isomorphism of Hilbert spaces $W^{2,k}(\mathbb{R}^n) \rightarrow W^{2,k-2}(\mathbb{R}^n)$ for all $k \geq 2$.