Exercise Sheet 13 - Solutions

1. Prove Lemma VII. 13.

   Solution: Let $T : D \to B_2$ be a bounded map and $v \in \overline{D}$. Pick a constant $C > 0$ such that $||Tw||_2 \leqslant C||w||_1$ for all $w \in D$. There exists a sequence $(v_n)_n \in D$ which converges to v.

   We put
   $$T_{\text{ext}}(v) := \lim_{n\to\infty} Tv_n.$$
   The limit exists because the sequence $(Tv_n)_n$ is Cauchy. This follows from the inequality
   $$||Tv_n - Tv_m||_2 \leqslant C||v_n - v_m||_1.$$
   Let $(v'_n)_n \in D$ be a sequence with limit v. Continuity of addition implies
   $$\lim_{n\to\infty} Tv_n - \lim_{n\to\infty} Tv'_n = \lim_{n\to\infty} \Big(Tv_n - Tv'_n\Big).$$
   Hence
   $$\Big|\Big|\lim_{n\to\infty} Tv_n - \lim_{n\to\infty} Tv'_n\Big|\Big|_2 \leqslant C \lim_{n\to\infty} ||v_n - v'_n||.$$
   Applying addition of continuity again implies $\lim_{n\to\infty} v_n - v'_n = 0$. This proves that $T_{\text{ext}}(v)$ does not depend on the chosen sequence $(v_n)_n$. Note that T_{ext} extends T because for any $v \in D$ the sequence $(v)_n$ converges to v and $(Tv)_n$ converges to Tv.

   Suppose $T' : \overline{D} \to B_2$ is a continuous extension of T. Then
   $$T'(v) = T'\Big(\lim_{n\to\infty} v_n\Big) = \lim_{n\to\infty} T'v_n \lim_{n\to\infty} Tv_n = T_{\text{ext}}(v_n).$$
   This implies $T' = T_{\text{ext}}$, so the extension is unique.

   Suppose $||Tw||_2 = ||w||_1$ for all $w \in D$. Then
   $$||T_{\text{ext}}(v)||_2 = ||\lim_{n\to\infty} Tv_n||_2 = \lim_{n\to\infty} ||Tv_n||_2 = \lim_{n\to\infty} ||v_n||_1 = ||v||_1.$$

2. Prove Lemma VII.17.

   Solution: Let $x \in D$ and $1 \leqslant i \leqslant n$. The space D equipped with the restriction of the Lebesgue measure is σ-finite because D is second-countable and locally compact. Set e_i to be the i'th vector in the standard basis. Fubini's theorem and the fundamental theorem of calculus imply

$$\begin{aligned}\frac{\partial F}{\partial x_i}(x) &= \lim_{\epsilon\to 0}\frac{F(x+\epsilon e_i)-F(x)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\int_Y \frac{f(x+\epsilon e_i,y)-f(x,y)}{\epsilon}d\mu(y)\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_Y\int_0^\epsilon \frac{\partial f}{\partial x_i}(x+te_i,y)dtd\mu(y)\\ &= \lim_{\epsilon\to 0}\int_0^\epsilon\frac{1}{\epsilon}\int_Y \frac{\partial f}{\partial x_i}(x+te_i,y)d\mu(y)dt\\ &= \lim_{\epsilon\to\infty}\frac{1}{\epsilon}\int_0^\epsilon G_i(x+te_i)dt.\end{aligned}$$

Because G_i is continuous, the limit on the last line converges to $G_i(x)$. The index i was arbitrary, so the gradient ∇F exists and is continuous because G_i is continuous. Thus $F \in C^1(D)$.

3. Prove the following formulae for all $f_1, f_2, f_3 \in L^1(\mathbb{R}^n)$

$$\begin{aligned} f_1 * f_2 &= f_2 * f_1,\\ (f_1 * f_2) * f_3 &= f_1 * (f_2 * f_3).\end{aligned}$$

Solution: Integration by substitution implies for all $x \in \mathbb{R}^n$ and $f_1, f_2 \in L^1(\mathbb{R}^n)$

$$f_1 * f_2(x) = \int_{\mathbb{R}^n} f_1(x-y)f_2(y)dy = \int_{\mathbb{R}^n} f_1(y)f_2(x-y)dy = f_2 * f_1(x).$$

Let $f, g, h \in L^1(\mathbb{R}^n)$. Integration by substitution gives for all $x \in \mathbb{R}^n$

$$((f*g)*h)(x) = \int_{\mathbb{R}^n}(f*g)(x-y)h(y)dy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y-z)g(z)h(y)dzdy.$$

Prop. VII. 15 allows us to apply Fubini's theorem for almost all $x \in \mathbb{R}^n$ to the above equality. This gives

$$((f*g)*h)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y-z)g(z)h(y)dydz = ((f*h)*g)(x).$$

This means we get $(f*g)*h = (f*h)*g$. We combine these formulae to get

$$(f_1 * f_2) * f_3 = (f_2 * f_1) * f_3 = (f_2 * f_3) * f_1 = f_1 * (f_2 * f_3)$$

for all $f_1, f_2, f_3 \in L^1(\mathbb{R}^n)$.

4. Let $||\cdot||_{2,k}$ be a Sobolev norm on $W^{2,k}(\mathbb{R}^n)$. Show that

$$||f|| := ||(1+||\xi||^k)\hat{f}||_{L^2}$$

defines a norm for $f \in W^{2,k}(\mathbb{R}^n)$ and that the norms $||\cdot||$ and $||\cdot||_{2,k}$ are equivalent on $W^{2,k}(\mathbb{R}^n)$.

Solution: The argument is based on the Plancherel formula. Let $f \in W^{2,k}(\mathbb{R}^n)$. Remark that for each each multiindex α with $|\alpha| \leqslant k$ Lemma VII. 32 and the Plancherel formula imply

$$||\partial^\alpha f||_2 = ||\xi^\alpha \widehat{f}||_2.$$

This implies that $||f||$ is well-defined. On the one hand, we estimate

$$\begin{aligned} ||f|| &= ||(1+||\xi||^k)\widehat{f}||_2 \\ &\leqslant ||f||_2 + \left(\int_{\mathbb{R}^n} \sum_{i=1}^n |\xi_i|^{2k} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \\ &= ||f||_2 + \left(\sum_{i=1}^n \left(\left(\int_{\mathbb{R}^n} |\xi_i^k \widehat{f}(\xi)|^2 d\xi\right)^{1/2}\right)^2\right)^{1/2} \\ &= ||f||_2 + \left(\sum_{i=1}^n \left|\left|\frac{\partial^k f}{\partial x_i^k}\right|\right|_2^2\right)^{1/2} \end{aligned}$$

For each vector $x \in \mathbb{R}^n$, we have $||x|| \lesssim ||x||_1$ because all norms are equivalent. We can apply this to the above inequality to get

$$||f|| \lesssim ||f||_2 + \sum_{i=1}^n \left|\left|\frac{\partial^k f}{\partial x_i^k}\right|\right|_2 \leqslant ||f||_{2,k}.$$

To get the inequality equality in the other direction requires more work. We have

$$\begin{aligned} ||f||_{2,k} &= \sum_{|\alpha|\leqslant k} ||\partial^\alpha f||_2 = \sum_{|\alpha|\leqslant k} ||\xi^\alpha \widehat{f}||_2 \\ &= \sum_{|\alpha|\leqslant k} \left(\int_{\mathbb{R}^n} |\xi^\alpha \widehat{f}(\xi)|^2 d\xi\right)^{1/2} \end{aligned}$$

The AM-GM inequality implies

$$x_1^{1/2} + \cdots + x_n^{1/2} \leqslant \sqrt{n}(x_1 + \cdots + x_n)^{1/2}$$

for all $x_1, \ldots, x_n \geqslant 0$ (square the inequality). In particular, we get

$$||f||_{2,k} \lesssim \left(\sum_{|\alpha|\leqslant k} \int_{\mathbb{R}^n} |\xi^\alpha \widehat{f}(\xi)|^2 d\xi\right)^{1/2}$$

Later, we will prove the inequality

$$\sum_{|\alpha|\leqslant k} |\xi^\alpha|^2 \lesssim (1+||\xi||^k)^2$$

for all $\xi \in \mathbb{R}^n$. We can plug this into the above inequality to get

$$||f||_{2,k} \lesssim \left(\int_{\mathbb{R}^n} |(1+||\xi||^k)\widehat{f}(\xi)|^2 d\xi \right)^{1/2} = ||f||.$$

We now turn to proving

$$\sum_{|\alpha| \leqslant k} |\xi^\alpha|^2 \lesssim (1+||\xi||^k)^2.$$

Note that

$$\sum_{|\alpha| \leqslant k} |\xi^\alpha|^2 \leqslant \left(\sum_{|\alpha| \leqslant k} |\xi^\alpha| \right)^2,$$

so it suffices to prove

$$\sum_{|\alpha| \leqslant k} |\xi^\alpha| \lesssim 1+||\xi||^k.$$

The right-hand side of this inequality does not vanish and both sides are continuous, so it suffices to prove the inequality for $||\xi|| \gg 1$. The binomial theorem implies

$$\sum_{|\alpha|=i} |\xi^\alpha| \leqslant ||\xi||_1^i$$

for all $i \geqslant 0$. If $||\xi||_1 > 1$ then

$$\sum_{|\alpha| \leqslant k} |\xi^\alpha| \lesssim \sum_{i=0}^{k} ||\xi||_1^i = \frac{||\xi||_1^{k+1}-1}{||\xi||_1 - 1} \lesssim ||\xi||_1^k \lesssim ||\xi||^k.$$

5. Let

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

be the Laplacian on $\mathbb{R}^n$. Show that $\Delta - \mathrm{id}\colon C_{00}^2(\mathbb{R}^n) \to C_{00}^2(\mathbb{R}^n)$ extends to an isomorphism of Hilbert spaces $W^{2,k}(\mathbb{R}^n) \to W^{2,k-2}(\mathbb{R}^n)$ for all $k \geqslant 2$.

Solution: Let $k \geqslant 2$. Lemma VII. 32 implies

$$((1-\Delta)f)^\wedge(\xi) = (1+||\xi||^2)\widehat{f}(\xi)$$

for all $f \in W^{2,k}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Note

$$1+||\xi||^k \lesssim (1+||\xi||^2)^{k/2} \lesssim 1+||\xi||^k$$

for all $\xi \in \mathbb{R}^n$. This follows from

$$\lim_{x \to +\infty} \frac{1+x^2}{(1+x)^2} = 1$$

. In particular, Exercise 4 implies that we can define the norm

$$||f||_{(k)} := ||(1 + ||\xi||^2)^{k/2}\widehat{f}||_2$$

for all $f \in W^{2,k}(\mathbb{R}^n)$ and this norm is equivalent to the Sobolev norm. The first remark implies

$$||(1-\Delta)f||_{(k-2)} = ||f||_{(k)}.$$

This implies that $1-\Delta$ is injective.

Let $f \in W^{2,k-2}(\mathbb{R}^n)$. Define

$$F := \left(\frac{\widehat{f}}{1+||\xi||^2}\right)^{\vee}.$$

This is well-defined because

$$\left|\left|\frac{\widehat{f}}{1+||\xi||^2}\right|\right|_2 \leqslant ||\widehat{f}||_2.$$

If we can prove $F \in W^{2,k}(\mathbb{R}^n)$, then Lemma VII. 32 implies $(1-\Delta)F = f$. In particular, $1-\Delta$ is surjective and hence an isomorphism (by the open mapping theorem).

We begin by proving the following claim: Let $g \in L^2(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$ such that $\xi^\alpha \widehat{g} \in L^2(\mathbb{R}^n)$. Then $((-i\xi)^\alpha \widehat{g})^\vee = \partial^\alpha g$. Indeed, let $\varphi \in C_{00}^\infty(\mathbb{R}^n)$ be a test function. The Plancherel formula (Theorem VII. 12) and Proposition VII. 5 imply

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)\overline{\partial^\alpha \varphi(x)}dx &= \int_{\mathbb{R}^n} \widehat{g}(\xi)(i\xi)^\alpha \overline{\widehat{\varphi}(\xi)}d\xi \\ &= \int_{\mathbb{R}^n} (i\xi)^\alpha \widehat{g}(\xi)\overline{\widehat{\varphi}(\xi)}d\xi \\ &= \int_{\mathbb{R}^n} ((i\xi)^\alpha \widehat{g})^\vee(x)\overline{(\widehat{\varphi})^\vee(x)}dx \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} ((-i\xi)^\alpha \widehat{g})^\vee(x)\overline{\varphi(x)}dx.\end{aligned}$$

This implies the claim by definition of a weak derivative.

Let $g\colon \mathbb{R}^n \to \mathbb{C}$ be a measurable function. One can modify the proof of Exercise 4 with the claim so that it implies $g \in W^{2,k}(\mathbb{R}^n)$ if and only if $||g|| < \infty$. Using the inequality at the beginning, this is satisfied if and only if $||g||_{(k)} < \infty$. Finally, we are in a position to prove $F \in W^{2,k}(\mathbb{R}^n)$ because Fourier inversion implies

$$||F||_{(k)} = ||f||_{(k-2)}.$$

The term on the right is finite because $f \in W^{2,k-2}(\mathbb{R}^n)$.