Exercise Sheet 1

1. Show Lemma I.3.

2. Let $\mathscr{H}$ be a Hilbert space with norm $||\cdot||$.
   (a) Prove: For all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $||x|| \leqslant 1$ and $||y|| \leqslant 1$ satisfy $||x-y|| > \epsilon$ then $||\frac{x+y}{2}|| < 1-\delta$. Compute δ as a function of ϵ.
   (b) Draw a picture of this geometric property.

3. Let $\mathscr{H}$ be a Hilbert space, $x, y, z \in \mathscr{H}$, $c \colon \mathbb{R} \to \mathscr{H}, t \mapsto tx + (1-t)y$ a parametrization of the line through x and y, and $f(t) := ||z - c(t)||^2$. Assuming $x \neq y$, show that f is strictly convex.

4. Let $C \subset \mathscr{H}$ be a closed, convex subset of a Hilbert space, and set $d(x, C) := \inf\{||x-y|| : y \in C\}$ for all $x \in \mathscr{H}$. Show that for each $x \in \mathscr{H}$, there is a unique point $p(x) \in C$ which satisfies $d(x, p(x)) = d(x, C)$.

   Hint: Let $(x_n)_n$ be a sequence in C with $d(x, x_n) \to d(x, C)$ as $n \to \infty$. Prove by using exercise 2, that any such sequence is Cauchy. Use exercise 3 to prove that any two points x_1, x_2 which satisfy $d(x, x_1) = d(x, x_2) = d(x, C)$ are equal.

5. Verify that $\bigwedge^\alpha(\mathbb{R})$ is a Banach space (see Example I.11).

6. Let $\alpha > 1$. Show that any $f \colon \mathbb{R} \to \mathbb{R}$ satisfying

$$\sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\alpha} < \infty$$

   is constant.

7. * One can define $\bigwedge^\alpha(X)$ for any metric space (X, d). Give a simple geometric condition on the metric space (X, d), which implies that for all $\alpha > 1$ the space $\bigwedge^\alpha(X)$ consists only of the constant functions.