Exercise Sheet 1 - Solutions

1. Show Lemma I.3.

   Solution: Note that the direct product $V_1 \times V_2$ of two normed spaces is again a normed space with the norm $||(v_1, v_2)|| := ||v_1||_{V_1} + ||v_2||_{V_2}$ (and this norm induces the product topology). Thus a sequence $((v_{1,n}, v_{2,n}))_n$ is Cauchy if and only if the sequences $(v_{1,n})_n$ and $(v_{2,n})_n$ are Cauchy.

   Let $v_1, v_2, w_1, w_2 \in V$. Then

   $$||(w_1 + w_2) - (v_1 + v_2)|| = ||(w_1 - v_1) + (w_2 - v_2)|| \leqslant ||w_1 - v_1|| + ||w_2 - v_2||$$

   by the triangle inequality. Thus for any Cauchy sequence $((v_{1,n}, v_{2,n}))_n$ the sequence given by adding all the vectors $(v_{1,n} + v_{2,n})_n$ is Cauchy. Thus addition is continuous.

   Let $\lambda_1, \lambda_2 \in \mathbb{K}$ and $v_1, v_2 \in V$. Then

   $$||\lambda_1 v_1 - \lambda_2 v_2|| = ||(\lambda_1 - \lambda_2)v_1 - \lambda_2(v_1 - v_2)|| \leqslant ||v_1|| \cdot |\lambda_1 - \lambda_2| + |\lambda_2| \cdot ||v_1 - v_2||.$$

   Using an argument similar to the one above, one can use this inequality to prove that multiplication is continuous.

2. Let $\mathscr{H}$ be a Hilbert space with norm $|| \cdot ||$.

   (a) Prove: For all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $||x|| \leqslant 1$ and $||y|| \leqslant 1$ satisfy $||x - y|| > \epsilon$ then $||\frac{x+y}{2}|| < 1 - \delta$. Compute δ as a function of ϵ.

   (b) Draw a picture of this geometric property.

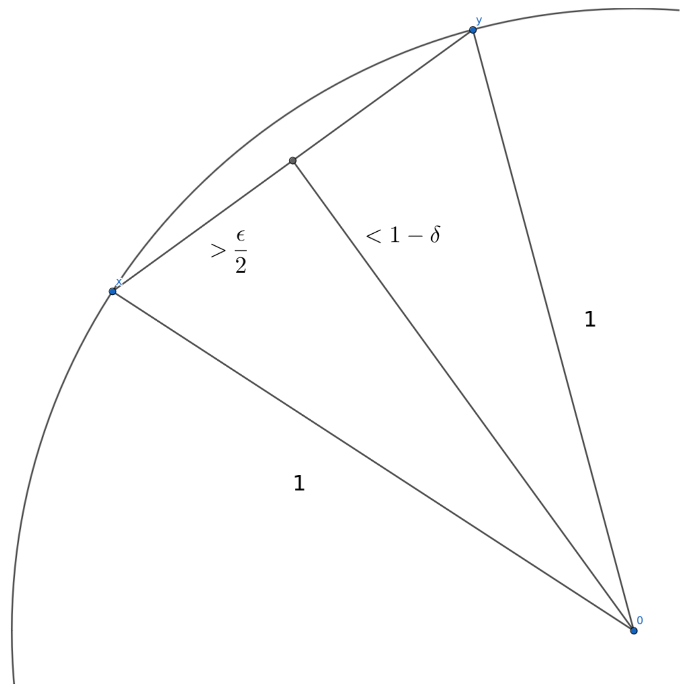
   Solution:

   (a) The parallelogram identity gives

   $$\left|\left|\frac{x+y}{2}\right|\right|^2 = (1/2)(||x||^2 + ||y||^2) - \left|\left|\frac{x-y}{2}\right|\right|^2 < 1 - \epsilon^2/4$$

   A possible choice is $\epsilon(\delta) = 1 - \sqrt{1 - \epsilon^2/4}$.

   (b)



3. Let $\mathscr{H}$ be a Hilbert space, $x, y, z \in \mathscr{H}$, $c\colon \mathbb{R} \to \mathscr{H}, t \mapsto tx + (1-t)y$ a parametrization of the line through x and y, and $f(t) := ||z - c(t)||^2$. Assuming $x \neq y$, show that f is strictly convex.

   Solution: The sesquilinearity of the inner product gives

   $$f(t) = ||z - y||^2 + 2t\mathrm{Re}\langle z - y, y - x\rangle + t^2||y - x||^2.$$

   Thus $f''(t) = 2||y - x||^2 > 0$ for all $t \in \mathbb{R}$, so f is strictly convex.

4. Let $C \subset \mathscr{H}$ be a closed, convex subset of a Hilbert space, and set $d(x, C) := \inf\{||x - y|| : y \in C\}$ for all $x \in \mathscr{H}$. Show that for each $x \in \mathscr{H}$, there is a unique point $p(x) \in C$ which satisfies $d(x, p(x)) = d(x, C)$.

   Hint: Let $(x_n)_n$ be a sequence in C with $d(x, x_n) \to d(x, C)$ as $n \to \infty$. Prove by using exercise 2, that any such sequence is Cauchy. Use exercise 3 to prove that any two points x_1, x_2 which satisfy $d(x, x_1) = d(x, x_2) = d(x, C)$ are equal.

   Solution: We first prove the existence of such a point. By shifting x and C if necessary, we are free to assume $x = 0$. Let $(x_n)_n$ be any sequence of vectors $x_n \in C$ with $d(0, x_n) \to d(0, C)$ as $n \to \infty$. Suppose this sequence is not Cauchy. Let $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exist $n, m \geqslant N$ with $||x_n - x_m|| > \epsilon$. Let $\delta > 0$ and $N > 0$ such that $d(0, x_n) - d(0, C) < \delta$

for all $n \geqslant N$. Pick $n, m \in \mathbb{N}$ with $||x_n - x_m|| > \epsilon$ and $n, m \geqslant N$. Because C is convex we have $d(0, C) \leqslant ||\frac{x_n + x_m}{2}||$. The parallelogram identity gives

$$d(0,C)^2 \leqslant \left|\left|\frac{x_n + x_m}{2}\right|\right|^2 = \frac{1}{2}(||x_n||^2 + ||x_m||^2) - \left|\left|\frac{x_n - x_m}{2}\right|\right|^2 \leqslant (d(0,C) + \delta)^2 - \epsilon^2/4.$$

By letting $\delta \to 0$, we arrive at $0 \leqslant -\epsilon^2/4$. This contradiction proves that the sequence is Cauchy.

Suppose there exist two distinct points $x_1, x_2 \in C$ with $d(x, x_1) = d(x, x_2) = d(x, C)$. We get

$$d\left(x, \frac{x_1 + x_2}{2}\right) < d(x, C)$$

from exercise 3. This contradicts the definition of $d(x, C)$ since $\frac{x_1 + x_2}{2} \in C$. Thus there can never be two distinct points satisfying this relation.

5. Verify that $\bigwedge^\alpha(\mathbb{R})$ is a Banach space (see Example I.11).

   Solution: We only prove that the space $\bigwedge^\alpha(\mathbb{R})$ is complete. Let $(f_n)_n$ be a Cauchy sequence in $\bigwedge^\alpha(\mathbb{R})$. For each $x \in \mathbb{R}$ the limit $\lim_{n\to\infty} f_n(x)$ exists because $|f_n(x) - f_m(x)| \leqslant ||f_n - f_m||_{\bigwedge^\alpha}$. Define the function $f(x) := \lim_{n\to\infty} f_n(x)$.

   We prove $f \in \bigwedge^\alpha(\mathbb{R})$. Because $(f_n)_n$ is a Cauchy sequence, there exists $C \geqslant 0$ with $||f_n||_{\bigwedge^\alpha} \leqslant C$ for all $n \in \mathbb{N}$. In particular, for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have $|f_n(x) - f_n(y)| \leqslant C|x - y|^\alpha$. Letting $n \to \infty$, we get the inequality $|f(x) - f(y)| \leqslant C|x - y|^\alpha$. By a similar argument, we get $|f(x)| \leqslant C$ for all $x \in \mathbb{R}$. Thus $f \in \bigwedge^\alpha(\mathbb{R})$.

   We prove $||f - f_n||_{\bigwedge^\alpha} \to 0$ as $n \to \infty$. Let $\epsilon > 0$. There exists $N > 0$ such that for all $n, m \geqslant N$ we have $||f_n - f_m||_{\bigwedge^\alpha} < \epsilon$. Consider $x, y, z \in \mathbb{R}$ with $y \neq z$, then there exists M such that for all $m \geqslant M$ we have $|f(x) - f_m(x)| < \epsilon$ and $|f(y) - f_m(y)| < \epsilon|y - z|^\alpha$ and $|f(z) - f_m(z)| < \epsilon|y - z|^\alpha$. For all $n \geqslant N$ and $m \geqslant \max(N, M)$ we get

$$\begin{aligned}|f(x) - f_n(x)| + \frac{|(f(y) - f_n(y)) - (f(z) - f_n(z))|}{|y - z|^\alpha} &\leqslant |f(x) - f_M(x)| + |f_M(x) - f_n(x)| \\ &+ \frac{|(f(y) - f_M(y)) - (f(z) - f_M(z))|}{|y - z|^\alpha} \\ &+ \frac{|(f_M(y) - f_n(y)) - (f_M(z) - f_n(z))|}{|y - z|^\alpha} \\ &< 5\epsilon.\end{aligned}$$

   Taking the supremum yields

$$||f - f_n||_{\bigwedge^\alpha} < 5\epsilon$$

   for all $n \geqslant N$. Hence $||f - f_n||_{\bigwedge^\alpha} \to 0$ as $n \to \infty$.

6. Let $\alpha > 1$. Show that any $f \colon \mathbb{R} \to \mathbb{R}$ satisfying

$$\sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\alpha} < \infty$$

is constant.

Solution: Let $C := \sup_{x_1 \neq x_2} \frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^\alpha}$ and $x < y$. Let $n \in \mathbb{N}$. Define the evenly spaced numbers $x = x_0 < x_1 < \cdots < x_n = y$. Then

$$|f(x) - f(y)| \leqslant \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| \leqslant \sum_{i=1}^{n} C(y-x)^\alpha n^{-\alpha} = C(y-x)^\alpha n^{1-\alpha}.$$

As $n \to \infty$, this inequality yields $f(x) = f(y)$. Hence f is constant.

7. * One can define $\bigwedge^\alpha(X)$ for any metric space (X, d). Give a simple geometric condition on the metric space (X, d), which implies that for all $\alpha > 1$ the space $\bigwedge^\alpha(X)$ consists only of the constant functions.