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Exercise Sheet 2

Let $(V, ||\cdot||)$ be a normed space.

1. Let $W \subset V$ be a subspace. For any $v \in V$ define $d(v, W) := \inf_{w \in W} ||v - w||$. Assume $W \subset V$ is closed and $W \neq V$. Show that for all $\epsilon > 0$ there exists $v \in V$ with $||v|| = 1$ and $d(v, W) > 1 - \epsilon$.

2. Let $W \subset V$ be a subspace. Define $||v + W|| := d(v, W)$ for all $v + W \in V/W$

   (a) Show that this defines a norm on V/W if and only if W is closed in V.

   (b) Shot that if V is Banach and W closed in V, then V/W is Banach.

   (c) Prove: If W is closed and $W \neq V$, then the canonical projection

   $$\pi \colon V \to V/W$$

   satisfies $||\pi|| = 1$

3. Construct an isometry $T \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ such that the image $R(T) \subset \ell^2(\mathbb{Z})$ is a closed, proper subspace of $\ell^2(\mathbb{Z})$.

4. Let $S := \{v \in V : \; ||v|| = 1\}$. Show that the following are equivalent:

   (a) $\dim(V) < +\infty$

   (b) S is compact.

   Hint: Use exercise 1 to prove (b) implies (a).