Exercise Sheet 2 - Solutions

Let $(V, ||\cdot||)$ be a normed space.

1. Let $W \subset V$ be a subspace. For any $v \in V$ define $d(v,W) := \inf_{w\in W} ||v-w||$. Assume $W \subset V$ is closed and $W \neq V$. Show that for all $\epsilon > 0$ there exists $v \in V$ with $||v|| = 1$ and $d(v,W) > 1-\epsilon$.

   Solution: Suppose there exists $\epsilon > 0$ such that $d(v,W) \leqslant ||v||(1-2\epsilon)$ for all $v \in V$. Let $v_0 \in V$. By assumption, there exists $w_1 \in W$ with $||v_0 - w_1|| \leqslant ||v_0||(1-\epsilon)$. Set $v_1 := v_0 - w_1$. Using recursion, we construct a sequence of vectors v_n such that $||v_n|| \leqslant (1-\epsilon)^n ||v_0||$ and $v_0 - v_n \in W$ for all $n \in \mathbb{N}$. Thus $v_0 = \lim_{n\to\infty}(v_0 - v_n)$ lies in W since W is closed. This argument works for any $v_0 \in V$, so $V = W$. This contradicts $V \neq W$.

2. Let $W \subset V$ be a subspace. Define $||v + W|| := d(v,W)$ for all $v + W \in V/W$

   (a) Show that this defines a norm on V/W if and only if W is closed in V.

   (b) Shot that if V is Banach and W closed in V, then V/W is Banach.

   (c) Prove: If W is closed and $W \neq V$, then the canonical projection

   $$\pi\colon V \to V/W$$

   satisfies $||\pi|| = 1$.

   Solution:

   (a) Suppose W is closed and consider $||v + W|| = 0$. There exists a sequence $w_n \in W$ with $||v - w_n|| < \frac{1}{n}$. The sequence $(w_n)_n$ is Cauchy with limit v. Thus $v \in V$ and hence $v + W = W$. The homogeneity and the triangle inequality follow from the corresponding properties of $||\cdot||$.

   Suppose $(w_n)_n$ is sequence in W with limit w. Then $||w+W|| \leqslant ||w - w_n||$ by definition for all $n \in \mathbb{N}$, so we get $||w + W|| = 0$ by letting $n \to \infty$. Since $||\cdot||_{V/W}$ is a norm, this implies $w + W = W$ or, equivalently, $w \in W$. Hence W is closed.

   (b) Consider a Cauchy sequence $(v_n + W)_n$ in V/W. There exists a subsequence $(v_{n_m} + W)_m$ with $||v_{n_m} - v_{n_{m+1}} + W|| < 2^{-m}$ for all m. We use recursion to pick representatives v_{n_m} with $||v_{n_m} - v_{n_{m+1}}|| < 2^{-m+1}$. Then $(v_{n_m})_n$ is Cauchy with some limit $v \in V$. Note that $||v + W - v_{n_m} + W|| \leqslant ||v - v_{n_m}||$, so $v + W$ is the limit of $(v_{n_m} + W)_n$. Thus $v + W$ is the limit of $(v_n + W)_n$ because this sequence is Cauchy. By exercise 2 (a), the space V/W is Banach.

   (c) We have $||v + W|| \leqslant ||v||$ for all $v \in V$, so $||\pi|| \leqslant 1$. By exercise 1, for all $\epsilon > 0$ there exists $v \in V$ with $||v|| = 1$ and $||v + W|| > 1 - \epsilon$. Thus $||\pi|| > 1 - \epsilon$ for all $\epsilon > 0$. By letting $\epsilon \to 0$, we arrive at $||\pi|| = 1$.

3. Construct an isometry $T\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ such that the image $R(T) \subset \ell^2(\mathbb{Z})$ is a closed, proper subspace of $\ell^2(\mathbb{Z})$.

Solution: Consider the linear map $T\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$(T((a_n)))_n := \begin{cases} a_{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ a_n & \text{if } n < 0. \end{cases}$$

It is an isometry because

$$||T((a_n)_n)||^2_{\ell^2} = \sum_n |T((a_n))_n||^2 = \sum_{n>0} |a_{n-1}|^2 + \sum_{n<0} |a_n|^2 = ||(a_n)||^2_{\ell^2}.$$

The image is $R(T) = \{(a_n) \in \ell^2(\mathbb{Z}) : a_0 = 0\}$, which is a closed subspace.

4. Let $S := \{v \in V : ||v|| = 1\}$. Show that the following are equivalent:

   (a) $\dim(V) < +\infty$

   (b) S is compact.

   Hint: Use exercise 1 to prove (b) implies (a).

   Solution: If $\dim(V) < +\infty$, the equivalence of all norms on finite-dimensional spaces implies that it is sufficient to prove that the sphere is compact. This follows from the Heine-Borel theorem.

   Suppose V is not finite-dimensional. By recursion and exercise 1, we can pick vectors v_n with $|v_n - v_m| > 1/2$ for all $m < n$ and $||v_n|| = 1$. No subsequence of this sequence can be Cauchy, so S is not compact.