Exercise Sheet 4

1. Show Prop. III.3 (1) and (2).

2. Let $\mathscr{H}$ be a separable Hilbert space and $\{e_i : i \geqslant 1\}$ an orthonormal basis for $\mathscr{H}$.

   (a) Prove: For all complex numbers $\lambda_i \in \mathbb{C}$, the operator

   $$T\colon \bigoplus_{i\geqslant 1} \mathbb{C}e_i \to \bigoplus_{i\geqslant 1} \mathbb{C}e_i,\ e_i \mapsto \lambda_i e_i$$

   extends to a bounded operator on $\mathscr{H}$ if and only if $\sup_i |\lambda_i| < \infty$.

   (b) Show that if T is compact, then $\lim_{i\to\infty} \lambda_i = 0$.

3. Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{B}_2(\mathscr{H})$ the space of Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm. Prove that $\mathscr{B}_2(\mathscr{H})$ is a Hilbert space.

4. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space such that $L^2(X, \mu)$ is separable.[1] Prove that every Hilbert-Schmidt operator on $L^2(X, \mu)$ is of the form T_K for some kernel $K \in L^2(X \times X, \mu \times \mu)$.

[1] See this mathoverflow thread for a discussion on when $L^2(X)$ is separable. More general sources of examples are Radon measures on second-countable LCH spaces, e.g. smooth measures on manifolds.