Exercise Sheet 5

1. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Show that
$$||TT^*|| = ||T^*T|| = ||T||^2.$$

2. Let $\mathscr{H}$ be a Hilbert space. An operator $T \in \mathscr{B}(H)$ is called normal if $TT^* = T^*T$. Show that if T is a compact normal operator, then $\mathscr{H}$ has an orthonormal basis of eigenvectors of T. Show that the eigenspaces satisfy $\dim_{\mathbb{K}}(\mathscr{H}_\lambda) < \infty$ for all $\lambda \neq 0$, and for all $\epsilon > 0$
$$|\{\lambda : |\lambda| \geqslant \epsilon,\ \mathscr{H}_\lambda \neq 0\}| < \infty.$$

3. Let $a > b > 0$. Set $I := (-a, a) \subset \mathbb{R}$ and $J := [-b, b] \subset I$. Let
$$C_b^1(I) := \{f \colon I \to \mathbb{R} : f \in C^1, ||f||_\infty < \infty, ||f'||_\infty \leqslant \infty\}.$$
This is a Banach space when equipped with the norm
$$||f|| := ||f||_\infty + ||f'||_\infty.$$
Let $C_b(J)$ be the space of continuous functions with norm $||\cdot||_\infty$. Prove that the restriction operator
$$C_b^1(I) \to C_b(J)$$
is compact.

   Hint: Use the Arzela-Ascoli theorem.

4. Let $\mathcal{B}$ be a Banach space of infinite dimension. Show that $\mathcal{B}$ does not admit a countable vector space basis.