Exercise Sheet 5 - Solutions

1. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Show that
$$||TT^*|| = ||T^*T|| = ||T||^2.$$

Solution: Note that we have $|\langle T^*Tv, v\rangle| = |\langle Tv, Tv\rangle| = ||Tv||^2$ for all $v \in \mathscr{H}$. Moreover, the operator T^*T is self-adjoint, so Lemma III. 16 implies
$$||T^*T|| = \sup\{||Tv||^2 : ||v|| = 1\} = (\sup\{||Tv|| : ||v|| = 1\})^2 = ||T||^2.$$
We find $|\langle TT^*v, v\rangle| = |\langle T^*v, T^*v\rangle| = ||T^*v||^2$ for all $v \in \mathscr{H}$. Now, the same argument as above shows $||TT^*|| = ||T^*||^2$. Corollary II. 11 says that we have $||T|| = ||T^*||$, so we can chain the above equalities to get
$$||TT^*|| = ||T^*||^2 = ||T||^2 = ||T^*T||.$$

2. Let $\mathscr{H}$ be a Hilbert space. An operator $T \in \mathscr{B}(H)$ is called normal if $TT^* = T^*T$. Show that if T is a compact normal operator, then $\mathscr{H}$ has an orthonormal basis of eigenvectors of T. Show that the eigenspaces satisfy $\dim_{\mathbb{K}}(\mathscr{H}_\lambda) < \infty$ for all $\lambda \neq 0$, and for all $\epsilon > 0$
$$|\{\lambda : |\lambda| \geqslant \epsilon,\ \mathscr{H}_\lambda \neq 0\}| < \infty.$$

Solution: Define the operators $S_1 := \frac{1}{2}(T + T^*)$ and $S_2 := \frac{1}{2}(T - T^*)$. Note that the adjoint of a compact operator is compact, so S_1 (and S_2) are compact.

Let $\mathscr{H}^1_\lambda$ be the eigenspace of S_1 to the eigenvalue $\lambda \in \mathbb{R}$. Note that this space is finite-dimensional because S_1 is compact. Because T is normal (and this is the crucial point!) we have $S_1S_2 = S_2S_1$. Let $v \in \mathscr{H}^1_\lambda$, then
$$S_1(S_2v) = S_2(S_1v) = \lambda(S_2v).$$
This implies $S_2v \in \mathscr{H}^1_\lambda$. Thus S_2 descends to an operator on $\mathscr{H}^1_\lambda$. The operator satisfies $(iS_2)^* = iS_2$, so the spectral theorem can be applied to $S_2|_{\mathscr{H}^1_\lambda}$ to give an orthonormal basis for $\mathscr{H}^1_\lambda$ consisting of eigenvectors of S_2. Pick such a basis B_λ for each λ.

The operator S_1 is self-adjoint, so by the spectral theorem the union of all the B_λ is an orthonormal basis of $\mathscr{H}$. Each vector in this basis is an eigenvector of S_1 and S_2, so this is an orthonormal basis of eigenvectors for $T = S_1 + S_2$.

Let $\mathscr{H}^2_{i\eta}$ be the eigenspace of S_2 of eigenvalue $i\eta$ for some $\eta \in \mathbb{R}$. Consider an eigenvector $v \in \mathscr{H}_{x+iy}$ to T with eigenvalue $x + iy$ with $x, y \in \mathbb{R}$. Then v is orthogonal to the eigenspaces $\mathscr{H}_\lambda$ with $\lambda \neq x + iy$, so v can be written in our basis as the sum of eigenvectors to the eigenvalues $x + iy$. Therefore $v \in \mathscr{H}^1_x \cap \mathscr{H}^2_{iy}$. This proves $\mathscr{H}_{x+iy} = \mathscr{H}^1_x \cap \mathscr{H}^2_{iy}$, so the space $\mathscr{H}_{x+iy}$ is finite-dimensional.

Let $\epsilon > 0$ and suppose there exist infinitely many non-zero eigenvalues $\lambda \in \mathbb{K}$ with $|\lambda| \geqslant \epsilon$. Let $v_n \in \mathscr{H}$ be an orthonormal sequence of eigenvectors, whose eigenvalues λ_n are pair-wise distinct and $|\lambda_n| \geqslant \epsilon$. Then the sequence $(Tv_n)_n$ can not have a converging subsequence. However, this contradicts the compactness of T.

3. Let $a > b > 0$. Set $I := (-a, a) \subset \mathbb{R}$ and $J := [-b, b] \subset I$. Let

$$C_b^1(I) := \{f : I \to \mathbb{R} : f \in C^1, ||f||_\infty < \infty, ||f'||_\infty \leqslant \infty\}.$$

This is a Banach space when equipped with the norm

$$||f|| := ||f||_\infty + ||f'||_\infty.$$

Let $C_b(J)$ be the space of continuous functions with norm $||\cdot||_\infty$. Prove that the restriction operator

$$C_b^1(I) \to C_b(J)$$

is compact.

Hint: Use the Arzela-Ascoli theorem.

Solution: Let $(f_n)_n$ be sequence in $C_b^1(I)$ with $||f_n|| \leqslant 1$ for all $n \geqslant 0$. We need to prove that the sequence $(f_n|_J)_n$ has a convergent subsequence. By the Arzela-Ascoli theorem, we need to show that the sequence is bounded from above and that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in J$ with $|x - y| < \delta$ we have $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$. We have $||f_n|| \leqslant 1$ so $|f_n(x)| \leqslant 1$ for all $n \in \mathbb{N}$ and $x \in J$. This proves that the sequence is bounded above.

The same argument proves $|f_n'(x)| \leqslant 1$ for all $x \in J$ and $n \in \mathbb{N}$. Let $x, y \in J$, then the mean value theorem implies that for each $n \in \mathbb{N}$ there exists a $\xi \in J$ such that

$$|f_n(x) - f_n(y)| = |f'(\xi)||x - y| \leqslant |x - y|.$$

So we can take $\delta = \epsilon$ to achieve equicontinuity.

4. Let $\mathcal{B}$ be a Banach space of infinite dimension. Show that $\mathcal{B}$ does not admit a countable vector space basis.

Solution: Suppose $\{v_i : i \geqslant 0\}$ is a countable basis of $\mathcal{B}$. Consider the subspaces $\mathcal{B}_n$ which are spanned by the vectors $v_1, \ldots, v_n$. We have $\bigcup_n \mathcal{B}_n = \mathcal{B}$ because $\{v_i : i \geqslant 0\}$ is a basis. On the other hand, the spaces $\mathcal{B}_n$ are closed subspaces because they are complete by the uniqueness of norms on finite-dimensional vector spaces. By the Baire category theorem, there exists $n \geqslant 0$ such that $\mathcal{B}_n$ has a non-empty interior. But this is impossible because any subspace with a non-empty interior is the entire space.