Exercise Sheet 6

1. Let E be a Banach space and $B^* \subset E^*$ a subset such that the set of values $\{f(x) : f \in B^*\}$ is bounded for each $x \in E$. Show that B^* is bounded.

2. Let E, F be Banach spaces and $f\colon E \times F \to \mathbb{K}$ a bilinear form such that

   (a) the map $x \in E \mapsto f(x,y) \in \mathbb{K}$ is continuous for all $y \in F$,

   (b) the map $y \in F \mapsto f(x,y) \in \mathbb{K}$ is continuous for all $x \in E$.

   Prove that there exists $C \geqslant 0$ such that $|f(x,y)| \leqslant C||x||||y||$ for all $x \in E$ and $y \in F$.

3. Assume V is a vector space endowed with two norms $||\cdot||_1$ and $||\cdot||_2$, such that $(V, ||\cdot||_1)$ and $(V, ||\cdot||_2)$ are Banach spaces. Suppose there exists $C \geqslant 0$ such that $||v||_1 \leqslant C||v||_2$ for all $v \in V$. Prove that there exists $K \geqslant 0$ such that $||v||_2 \leqslant K||v||_1$ for all $v \in V$.

4. Let $C([0,1])$ and $C^1([0,1])$ both be endowed with $||f||_\infty := \sup_{x\in[0,1]} |f(x)|$. Show that the derivative

$$\begin{aligned} C^1([0,1]) &\to C([0,1]) \\ f &\mapsto f' \end{aligned}$$

   is an unbounded operator, but has a closed graph.