Exercise Sheet 6 - Solutions

1. Let E be a Banach space and $B^* \subset E^*$ a subset such that the set of values $\{f(x) : f \in B^*\}$ is bounded for each $x \in E$. Show that B^* is bounded.

   Solution: The principle of uniform boundedness implies the existence of a constant $C \geqslant 0$ such that
   $$\sup_{f \in B^*} \sup_{x \in B_{\leqslant 1}(0)} |f(x)| \leqslant C.$$
   In other words, we get
   $$\sup_{f \in B^*} ||f|| \leqslant C$$
   which means that B^* is bounded.

2. Let E, F be Banach spaces and $f \colon E \times F \to \mathbb{K}$ a bilinear form such that

   (a) the map $x \in E \mapsto f(x, y) \in \mathbb{K}$ is continuous for all $y \in F$,

   (b) the map $y \in F \mapsto f(x, y) \in \mathbb{K}$ is continuous for all $x \in E$.

   Prove that there exists $C \geqslant 0$ such that $|f(x, y)| \leqslant C||x||||y||$ for all $x \in E$ and $y \in F$.

   Solution: We give two solutions. Consider the set $B^* = \{f(x, \cdot) : ||x|| \leqslant 1\} \subset E^*$. By the second assumption the set $\{f(x, y) : ||x|| \leqslant 1\}$ is bounded for each $y \in F$. By exercise 1, there exists a constant C such that $||f(x, \cdot)|| \leqslant C$ for all $||x|| \leqslant 1$. This implies $|f(x, y)| \leqslant C||x||||y||$ by definition of the operator norm.

   Here is another solution using the closed graph theorem. For each $x \in E$ define the functional $f_x(y) := f(x, y)$. The statement of the exercise is equivalent to proving that the map
   $$x \in E \mapsto f_x \in F^*$$
   is continuous. Consider a Cauchy sequence $(x_n, f_{x_n}) \in E \times F^*$ with limit (x, g). Then we have
   $$g(y) = \lim_{n \to \infty} f(x_n, y) = f(\lim_{n \to \infty} x_n, y) = f(x, y).$$
   for all $y \in Y$. This implies that the graph of the mapping $x \in E \mapsto f_x \in F^*$ is closed, so the map must be continuous by the closed graph theorem.

3. Assume V is a vector space endowed with two norms $|| \cdot ||_1$ and $|| \cdot ||_2$, such that $(V, || \cdot ||_1)$ and $(V, || \cdot ||_2)$ are Banach spaces. Suppose there exists $C \geqslant 0$ such that $||v||_1 \leqslant C||v||_2$ for all $v \in V$. Prove that there exists $K \geqslant 0$ such that $||v||_2 \leqslant K||v||_1$ for all $v \in V$.

   Solution: The assumption implies that the mapping $(V, || \cdot ||_1) \to (V, || \cdot ||_2)$ is continuous. By the open mapping theorem, the inverse $(V, || \cdot_2 ||) \to (V, || \cdot_1 ||)$ is continuous. Thus there exists a constant $K > 0$ with $||v||_2 \leqslant K||v||_1$ for all $v \in V$.

4. Let $C([0,1])$ and $C^1([0,1])$ both be endowed with $||f||_\infty := \sup_{x\in[0,1]} |f(x)|$. Show that the derivative

$$\begin{aligned} C^1([0,1]) &\to C([0,1]) \\ f &\mapsto f' \end{aligned}$$

is an unbounded operator, but has a closed graph.

Solution: Let $n \in \mathbb{N}$ and consider the function $f(x) = x^n$. This function satisfies $||f|| = 1$ but $||f'|| = n$, so the derivative can not be a bounded operator.

Consider a Cauchy sequence (f_n, f_n') in the graph with limit (f, g). To prove the graph is closed means to prove $f' = g$. Note that $\lim_{n\to\infty} ||f_n' - g|| = 0$ by assumption. Let $x \in \mathbb{R}$ then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \lim_{n\to\infty} \frac{f_n(x+h) - f_n(x)}{h} \end{aligned}$$

For each $n \in \mathbb{N}$ and small enough $h > 0$ the mean value theorem shows that there is $\xi \in (x, x+h)$ with $f_n(x+h) - f_n(x) = hf_n'(\xi)$. This implies

$$\frac{f_n(x+h) - f_n(x)}{h} = f_n'(\xi) = (f_n'(\xi) - g(\xi)) + (g(\xi) - g(x)) + g(x) = g(x) + o(1)$$

as $h \to 0$. Combining the formulas yields

$$\lim_{h\to 0} \lim_{n\to\infty} \frac{f_n(x+h) - f_n(x)}{h} = g(x).$$