Exercise Sheet 7

1. Let $E \subset V$ be a closed subspace of a Banach space V. Prove: There exists a closed complement to E if and only if there exists a continuous linear map $P\colon V \to V$ with $P^2 = P$ and $\operatorname{im}(P) = E$.

2. Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ be Banach spaces and $T\colon V \to W$ a surjective, linear, and continuous map. Show that the following are equivalent:

   (a) The closed subspace $\ker(T)$ admits a closed complement in V.

   (b) There is a linear, continuous map $S\colon W \to V$ with $T \circ S = \mathrm{id}_W$.

3. Show that the subspaces

$$V := \{f \in \ell^1(\mathbb{N}) : f(2n) = 0\ \forall n \geqslant 0\}$$
$$W := \{f \in \ell^1(\mathbb{N}) : f(2n-1) = nf(2n)\ \forall n \geqslant 1\}$$

   are closed in $\ell^1(\mathbb{N})$ while $V + W$ is not closed.

   Hint: Show $V + W \supset c_{00}(\mathbb{N})$.

4. Show that there is a bounded set function $p\colon \mathscr{P}(\mathbb{N}) \to \mathbb{R}$ such that

   (a) $p(\mathbb{N}) = 1$,

   (b) $p(A \cup B) = p(A) + p(B)$ whenever $A \cap B$ is finite.