Exercise Sheet 9 - Solutions

1. Let $T\colon V \to W$ be a bounded linear operator of normed spaces. Prove that the adjoint $T^*\colon W^* \to V^*$ is continuous in the weak*-topology on W^* and V^*.

   Solution: Define the evaluation map $\mathrm{ev}_v(f) := f(v)$ for each $v \in V$ or $v \in W$. Let $v \in V$, then, by Lemma V. 19, we need to prove that the map $\mathrm{ev}_v \circ T^*$ is weakly continuous. We have
$$(\mathrm{ev}_v \circ T^*)(f) = (T^* f)(v) = f(Tv) = \mathrm{ev}_{Tv}(f).$$
   So the map $\mathrm{ev}_v \circ T^*$ is an evaluation map and hence continuous in the weak*-topology.

2. Let V be a normed space. Assume that $f\colon V \to \mathbb{K}$ is a linear form that is continuous w.r.t. the weak topology on V. Show that f is strongly continuous.

   Solution: We give two solutions to this exercise. By Proposition V.11, there exist $f_1, \dots, f_n \in V^*$ such that if $|f_i(v)| \leqslant 1$ then $|f(v)| \leqslant 1$. If $v \in V$ satisfies $f_i(v) \neq 0$ for some f_i then this implies
$$|f(v)| \leqslant |f_1(v)| + \cdots + |f_n(v)|.$$
   Let $v \in V$ such that $f_i(v) = 0$ for all f_i. We have $f_i(\lambda v) = 0$ for each $\lambda \in \mathbb{R}$ and hence $|f(\lambda v)| \leqslant 1$. This implies $f(v) = 0$, so we get the inequality
$$|f(v)| \leqslant |f_1(v)| + \cdots + |f_n(v)|.$$
   for all $v \in V$. This implies
$$|f(v)| \leqslant (||f_1|| + \cdots + ||f_n||)||v||$$
   for each $v \in V$.

   The second solution goes as follows. Let $U \subset \mathbb{R}$ be open, then the set $f^{-1}(U)$ is weakly open because f is weakly continuous. The weak topology on V is coarser than the strong topology, so the set $f^{-1}(U)$ is open. Because U was arbitrary this implies that f is continuous.

3. For each $n \geqslant 1$ define
$$\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{i/n}.$$
   What is the weak*-limit of the sequence μ_n as $n \to \infty$?

   Solution: Let $f \in C([0,1])$. Because f is continuous it is Riemann integrable. In particular, we have
$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(i/n).$$
   So the weak*-limit of μ_n is the integral $f \mapsto \int_0^1 f(x)dx$.

4. Let V be a normed space. Show that any linear form $\lambda\colon V^* \to \mathbb{K}$ that is continuous with respect to the weak*-topology is of the form $\lambda(f) = f(v)$ for some $v \in V$.

   Hint: The solution is similar to the solution of Exercise 2.

   Solution: There exist $v_1, \dots, v_n \in V$ such that

$$|\lambda(f)| \leqslant |f(v_1)| + \cdots + |f(v_n)|$$

for all $f \in V^*$ by the same argument as in Exercise 2. So the functional λ vanishes on the strongly closed subspace

$$W := \{f \in V^* | f(v_i) = 0\}.$$

Let $U := \langle v_1, \dots, v_n \rangle \subset V$ and denote by $i\colon U \to V$ the inclusion. The kernel of i^* is W and i^* is surjective. Because the functional λ vanishes on W, this implies that we can factor λ as

$$V^* \xrightarrow{i^*} U^* \xrightarrow{\overline{\lambda}} \mathbb{K}$$

Thus the adjoint λ^* factors as

$$\mathbb{K} \xrightarrow{\overline{\lambda}^*} U^{**} \xrightarrow{i^{**}} V^{**}$$

Because U is finite-dimensional, there exists $u \in U$ with $\overline{\lambda}(1) = \mathrm{ev}_u$. This implies

$$\lambda = \lambda^*(1) = i^{**}(\mathrm{ev}_u) = \mathrm{ev}_{i(u)}.$$