The geometric form of Hahn – Banach for real vector spaces will be used in the theory of topological vector spaces, in particular to establish the Krein – Milman theorem.

For many applications to dual space of normed $\mathbb{K}$-vector spaces, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the notion of seminorm, a bit more restrictive than gauge, will suffice.

<u>Def II. 6</u> : A seminorm on a $\mathbb{K}$-vector space V is a function $p : V \longrightarrow [0, +\infty[$ such that

(1) $p(\alpha \cdot v) = |\alpha| \cdot p(v) \quad \forall \alpha \in \mathbb{K}, \ \forall v \in V$

(2) $p(v_1 + v_2) \leq p(v_1) + p(v_2) . \quad \forall v_1, v_2 \in V.$

We have then the following form of Hahn–Banach valid for $\mathbb{K}$-vector spaces.

<u>Thm II.7</u>: Let V be a $\mathbb{K}$-vector space, $p: V \to [0, +\infty[$ a semi-norm, $M \subset V$ a $\mathbb{K}$-vector subspace and $f: M \to \mathbb{K}$ a linear form with $|f(v)| \leq p(v)$ $\forall v \in M$. Then there is a $\mathbb{K}$-linear extension $F: V \to \mathbb{K}$ with $|F(v)| \leq p(v)$ $\forall v \in V$.

<u>Proof</u>: We can assume $\mathbb{K} = \mathbb{C}$, since for $\mathbb{K} = \mathbb{R}$, Thm II.7 follows from Thm II.4.

Now let $f_1(v) = \mathrm{Re}\, f(v)$ $\forall v \in M$.

Then $|f_1(v)| \leq p(v)$ and there is an extension $F_1: V \to \mathbb{R}$ with

$$|F_1(v)| \leq p(v) \quad \forall v \in V.$$

Now define $F(v) := F_1(v) - i\, F_1(i \cdot v)$

Then F is $\mathbb{C}$ – linear and extends f. Finally we have to show that $|F|$ is bounded by p. Let $v \in V$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that:

$$|F(v)| = \alpha \cdot F(v) = F(\alpha \cdot v) = F_1(\alpha \cdot v)$$

Thus $|F(v)| = F_1(\alpha \cdot v) \leq p(\alpha \cdot v) = p(v)$.

$\boxed{\wr}$

We draw some immediate consequences which can loosely be summarized by saying that a normed $\mathbb{K}$-vector space has enough continuous linear functionals.

<u>Corollary II.8</u>

Let $(V, \|\cdot\|)$ be a normed $\mathbb{K}$-vector space, $M \subset V$ a subspace and $f : M \to \mathbb{K}$ continuous linear. Then there is

$$F : V \to \mathbb{K}$$

continuous linear with $F|_M = f$ and $\|F\| = \|f\|$.

<u>Proof:</u> By hypothesis $|f(v)| \leq \|f\| \cdot \|v\|$ $\forall v \in M$ where $\|f\| = \sup\limits_{\substack{\|w\| \leq 1 \\ w \in M}} |f(w)|$

Let $p : V \to [0, +\infty[$ be defined by

$$p(v) = \|f\| \cdot \|v\|.$$

Then p is even a norm and Thm II.7 applies to obtain a $\mathbb{K}$-linear form

$$F : V \to \mathbb{K}$$

with $F|_M = f$ and $|F(v)| \leq p(v) = \|f\| \cdot \|v\|$ $\forall v \in V$. This implies that

$$\|f\| = \sup_{\substack{v \in M \\ \|v\| \leq 1}} |f(v)| = \sup_{\substack{v \in M \\ \|v\| \leq 1}} |F(v)| \leq \sup_{\substack{v \in V \\ \|v\| \leq 1}} |F(v)| \leq \|f\|$$

and shows that $\|F\| = \|f\|$. $\boxed{\xi}$

<u>Corollary II. 9</u>

Let $(V, \|\cdot\|)$ be a normed space and $v_0 \in V$. Then there $\exists$ $f_0 \in V^*$ with

(1) $\|f_0\| = 1$

(2) $f_0(x_0) = \|x_0\|$

<u>Proof:</u> Let $M = \mathbb{K}\cdot x_0$ and $f : M \to \mathbb{K}$

$f(t\cdot x_0) = t\cdot \|x_0\|$. By corollary II.8

there is $f_0 \in V^*$ with $f_0\big|_M = f$,

in particular $f_0(x_0) = \|x_0\|$ and

$$\|f_0\| = \|f\| = \sup_{|t| \leq \frac{1}{\|x_0\|}} |f(t\cdot x_0)| = 1.$$

$\boxed{\exists}$

The following is an immediate consequence of II.9:

<u>Corollary II.10</u>

Let $(V, \|\ \|)$ be a normed space, then

$\forall v \in V$:

$$\|v\| = \sup\{|f(v)| : f \in V^*, \|f\| \leq 1\}$$

$$= \max\{|f(v)| : f \in V^*, \|f\| \leq 1\}.$$

This corollary allows us now to compute

the norm of the adjoint $T^* : W^* \to V^*$

of a bounded linear map $T : V \to W$.

Corollary II.11 We have $\|T^*\| = \|T\|$.

Proof:

$$\|T^*\| = \sup_{\substack{\|\lambda\| \le 1 \\ \lambda \in W^*}} \|T^*(\lambda)\|$$

$$= \sup_{\|\lambda\| \le 1} \sup_{\|v\| \le 1} \underbrace{|T^*(\lambda)(v)|}_{|\lambda(T(v))|}$$

$$= \sup_{\|v\| \le 1} \sup_{\|\lambda\| \le 1} |\lambda(T(v))|$$

But by Cor. II.10 we have:

$$\sup_{\|\lambda\| \le 1} |\lambda(T(v))| = \|T(v)\|$$

which with the above implies

$$\|T^*\| = \sup_{\|v\| \le 1} \|T(v)\| = \|T\|.$$

$\boxed{\S}$

Another application of II.10 is to the bidual of a normed space V: this is by definition $V^{**} = \mathcal{B}(V^*, \mathbb{K})$ and the point is that we have a canonical map $J : V \longrightarrow V^{**}$ defined by

$$J(v)(\lambda) := \lambda(v).$$

<u>Proposition II.12:</u>

The map $J : V \longrightarrow V^{**}$ is a $\mathbb{K}$-linear isometry into the Banach space V^{**}.

<u>Proof</u>: First, we have that

$$|J(v)(\lambda)| = |\lambda(v)| \leq \|\lambda\| \cdot \|v\|$$

which shows that $J(v) \in (V^*)^*$.

Next we have by II.10:

$$\|J(v)\| = \sup_{\|\lambda\| \leq 1} |\lambda(v)| = \|v\|.$$

$\boxed{\Sigma}$

Spaces for which J is surjective are called reflexive, they are automatically Banach spaces.

Now we turn to some important examples of dual spaces.

<u>Example II.13</u> [The dual of $L^p(X)$, $1 \leq p < +\infty$].

Let $1 \leq p < +\infty$ and q the conjugate exponent that is

$$\frac{1}{p} + \frac{1}{q} = 1 .$$

Then Hölder's inequality shows that every $g \in L^q(X)$ gives rise to a continuous linear functional on $L^p(X)$ by

$$\ell_g(f) := \int_X f(x) g(x) \, d\mu(x)$$

with $\|\ell\| \leq \|g\|_q$. In fact

<u>Theorem II. 14</u> $1 \leq p < +\infty$.

The map
$$L^q(X) \longrightarrow (L^p(X))^*$$
$$g \longmapsto \ell_g$$
is an isometric isomorphism.

See Stein–Shakarchi Chapter 1 §4 for a proof using the Radon-Nykodim theorem.

For a different proof see Struwe Chapter 4, §4. For $1 < p < +\infty$ this proof uses Clarkson's inequalities which imply that for these values of p, $L^p(X)$ is uniformly convex.

For $2 \leq p < +\infty$ the statement is:

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right)$$

$$\forall f, g \in L^p(X)$$

For $1 < p \leq 2$:

$$\left\| \frac{f+g}{2} \right\|_p^q + \left\| \frac{f-g}{2} \right\|_p^q \leq \frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right)^{q/p}$$

These are substitutes of the parallelogramm identity in Hilbert spaces ; for $2 \leq p < +\infty$ the proof is elementary while for $1 < p \leq 2$ the proof is much trickier!

An immediate Corollary is

<u>Corollary II.15</u> For $1 < p < +\infty$, $L^p(X)$ is reflexive.

We will see later on that for Banach spaces there is a relation between uniform convexity and reflexivity.

<u>Example II.16</u> Let X be a locally compact Hausdorff space. A continuous function $f : X \longrightarrow \mathbb{R}$ is said to vanish at infinity if $\forall \varepsilon > 0$ $\exists K \subset X$ compact such that

$$|f(x)| < \varepsilon \quad \forall x \in X \setminus K .$$

Let $C_0(X, \mathbb{C}) = \{ f : X \longrightarrow \mathbb{C}$, continuous vanishing at $+\infty \}$.

Endowed with the norm $\|f\|_0 := \sup_{x \in X} |f(x)|$

$C_0(X, \mathbb{C})$ is a Banach space.

Then the dual space $C_0(X, \mathbb{C})^*$ is described by the space of complex measures:

a complex measure is a set function

$$\mu : \mathcal{B}_X \longrightarrow \mathbb{C}$$

defined on the σ-algebra $\mathcal{B}_X$ of Borel sets such that for all $E \in \mathcal{B}_X$ and any countable partition $E = \bigsqcup_{i \in \mathbb{N}} E_i$ with $E_i \in \mathcal{B}_X$ we have

$$\mu(E) = \sum_{i=1}^{n} \mu(E_i) .$$

Since this series is assumed to converge for any permutation of the summands, it converges absolutely. One defines then the total variation of μ:

$$|\mu|(E) = \sup \Big\{ \sum_{i=1}^{\infty} |\mu(E_i)| : \; E = \bigsqcup_{i \in \mathbb{N}} E_i , \; E_i \in \mathcal{B}_X \Big\}$$

and shows that $|\mu|$ is a positive measure on $\mathcal{B}_X$ with $|\mu|(X) < +\infty$.

In order to define the integral of say a bounded Borel function $f : X \to \mathbb{C}$ wrt μ, one reduces oneself to the case of positive measures (where the Lebesgue integral is available) in the following way.

First one evidently can decompose μ as

$$\mu = \mu_1 + i \cdot \mu_2$$

where μ_i are both complex measures with values in $\mathbb{R}$: such measures are called <u>signed</u> measures. Given a signed measure $\lambda : \mathcal{B}_X \to \mathbb{R}$ define then

$$\lambda^+ = \frac{1}{2}(|\lambda| + \lambda)$$

$$\lambda^- = \frac{1}{2}(|\lambda| - \lambda)$$

Then λ^+, λ^- are positive measures, with $\lambda^+(X) < +\infty$, $\lambda^-(X) < +\infty$ and

$$\lambda = \lambda^+ - \lambda^-.$$

Thus given a complex measure μ, we can decompose it as follows into a combination of positive measures:

$$\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$$

and hence $\int_X f \, d\mu$ makes sense for say any bounded Borel function. Finally, we say that μ is regular if its total variation measure $|\mu|$ is a regular Borel measure. Then:

<u>Thm II. 17</u> (Riesz Representation) For every bounded linear map $\Phi : C_0(X, \mathbb{C}) \to \mathbb{C}$ there is a unique complex regular measure μ defined on Borel sets such that

$$\Phi(f) = \int_X f \, d\mu \quad \forall f \in C_0(X, \mathbb{C})$$

In addition $\|\Phi\| = |\mu|(X)$.

Details can be found in Rudin "Real and complex Analysis" Chapter 6 or FA II Chapter 8.

The Problem of Measure.

The Hahn-Banach theorem can be used to show that there is a finitely additive set function defined on all subsets of $\mathbb{R}^d$ that agrees with Lebesgue measure on measurable sets and is translation invariant. However this set function cannot be σ-additive and this is connected to the existence of non-measurable sets.

A deeper fact is that it is not possible to extend Lebesgue measure on $\mathbb{R}^d$, $d \geq 3$, as a finitely-additive measure on all subsets of $\mathbb{R}^d$ so that it both translation and rotation invariant!