V. Topological vector spaces, weak topologies, and the Banach-Alaoglu theorem.

In Analysis one encounters function spaces with a natural topology that however cannot be described by a single norm: for example the space of continuous functions on $\mathbb{R}$ with the topology of uniform convergence on compact sets. Another problem one encounters is the fact that the unit ball in an infinite dimensional Banach space is never compact.

To remediate these problems we are going to study topological vector spaces whose topology is given by a family of semi-norms. On one hand this allows us to study natural function spaces with the tools of functional analysis and on the other hand will lead to weaker topologies on Banach spaces thereby restoring compactness in certain situations.

V. 1. Basic definitions and Examples.

We begin by recalling Def. I.4 :

Def. V.1 A topological vector space is a $\mathbb{K}$-vector space V endowed with a topology such that the maps :

$$(1)\quad \mathbb{K} \times V \longrightarrow V$$
$$(\lambda, v) \longmapsto \lambda \cdot v$$

and

$$(2)\quad V \times V \longrightarrow V$$
$$(v, w) \longmapsto v + w$$

are continuous.

Let's draw the following very usefull conclusions : let

$$M_\lambda : V \longrightarrow V, \quad \lambda \in \mathbb{K}$$
$$v \longmapsto \lambda \cdot v$$

and

$$L_v : V \longrightarrow V$$
$$w \longmapsto v + w.$$

then

<u>Lemma V.2.</u> $\forall \lambda \neq 0$, $M_\lambda : V \to V$ and $\forall u \in V$, $L_u : V \to V$ are homeomorphisms.

<u>Proof:</u> M_λ is continuous with continuous left and right inverse $M_{\lambda^{-1}}$; L_u is continuous with continuous left and right inverse L_{-u}. $\boxed{\text{☰}}$

We now turn to describe the topology on a $\mathbb{K}$-vector space V generated by a family of semi-norms (see Def. II.6).

Let V be a $\mathbb{K}$-vector space and $\{\| \ \|_\alpha : \alpha \in A\}$ a family of seminorms

$$\| \ \|_\alpha : V \to [0, +\infty[$$

on V. There is a priori no restriction on the cardinality of A.

$\forall v \in V$, $F \subset A$ <u>finite</u>, and $r > 0$

let

$$N(v; F; r) = \{ w \in V : \| w - v \|_\alpha < r \quad \forall \alpha \in F \}$$

<u>Def. IV.3</u>:

Define $\Omega \subset V$ to be <u>open</u> if $\forall v \in \Omega$ $\exists$ $F \subset A$ finite and $r > 0$ such that

$$N(v; F; r) \subset \Omega .$$

Then ϕ and V are open and it is also clear that an arbitrary union of open sets is open. As for ~~the~~ finite intersections observe first that

$$N(v; F; r) = \bigcap_{\alpha \in F} N(v; \alpha; r) .$$

Next let $v_1, v_2 \in V$ and

$$v_3 \in N(v_1; \alpha_1; r_1) \cap N(v_2; \alpha_2; r_2) .$$

Then ~~with~~ we have: $\|v_3 - v_1\|_{\alpha_1} < r_1$ and $\|v_3 - v_2\|_{\alpha_2} < r_2$. Let

$$\varepsilon = \min\left(r_1 - \|v_3 - v_1\|_{\alpha_1},\ r_2 - \|v_3 - v_2\|_{\alpha_2}\right)$$

Then it follows from the triangle inequality:

$$N(v_3 ; \{\alpha_1, \alpha_2\} ; \varepsilon) \subset N(v_1 ; \alpha_1 ; r_1) \cap N(v_2 ; \alpha_2 ; r_2).$$

This implies that a finite intersection of open subsets is open.

<u>Def. IV. 4</u> The topology on V generated by the family of semi-norms $\{\|\ \|_\alpha : \alpha \in A\}$ is the topology whose open subsets are given by Def. IV. 3.

<u>Lemma V.5</u> The topology on V generated by the family of seminorms $\{\|\ \|_\alpha : \alpha \in A\}$ endows V with the structure of a topological vector space.

<u>Proof</u>: This follows from

$$(1) \quad N(\lambda \cdot v; F; |\lambda| \cdot r) = \lambda \cdot N(v; F; r)$$

and

$$(2) \quad N(v_1; F_1; r_1) + N(v_2; F_2; r_2)$$

$$\subset N(v_1 + v_2; F_1 \cap F_2; r_1 + r_2). \qquad \boxed{\S}$$

Of course, if $A = \phi$ then the topology on V has two open sets, namely ϕ and V. The following property keeps degenerate cases away:

<u>Def. V. 6.</u> A family A of seminorms on V is <u>sufficient</u> if $\forall v \in V$, $v \neq 0$ there is $\alpha \in A$ with $\|v\|_\alpha \neq 0$.

<u>Lemma V. 7</u> If A is sufficient then the topology generated by A is Hausdorff.

<u>Proof</u>: By using translations we are reduced to the case where we need to separate 0 from $v \neq 0$. Pick $\alpha \in A$ with $\|v\|_\alpha \neq 0$; let $\varepsilon = \|v\|_\alpha > 0$.

Then $N(0; \alpha; \frac{\varepsilon}{3}) \cap N(v; \alpha; \varepsilon/3) = \phi$.

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A particularly important case is when we have a $\underbrace{\text{countable family}}_{\text{sufficient}}$ of seminorms.

<u>Prop. V.8</u> If $\{\| \ \|_n : n \in \mathbb{N}_{\geq 1}\}$ is a sufficient countable family of seminorms on V, the generated topology is metrizable.

<u>Proof</u>: Since the family is sufficient one verifies easily that

$$d(v,w) := \sum_{n=1}^{\infty} \left(\frac{\|v-w\|_n}{1+\|v-w\|_n} \right) \frac{1}{2^n}$$

defines a distance on V. Then observe that $\forall n \geq 1$ and $\ell \geq 1$:

$$\frac{1}{2^n}\left(\frac{\|v-w\|_n}{1+\|v-w\|_n}\right) \leq d(v,w) \leq \sum_{k=1}^{\ell} \left(\frac{\|v-w\|_k}{1+\|v-w\|_k}\right)\frac{1}{2^k} + \frac{1}{2^\ell}.$$

From the second inequality we deduce that if $\frac{1}{2^\ell} < \varepsilon$ then:

$$N(v;\{\|\ \|_1,\dots,\|\ \|_e\};\varepsilon)\subset B_{<2\varepsilon}(v)$$

(where $B_{<r}(v)$ refers to balls for the distance d);

Let $\varepsilon \leqslant 1$ and $d(v,w) < \frac{\varepsilon}{2^{n+1}}$.

Then the first inequality implies

$$\frac{1}{2^n}\left(\frac{\|v-w\|_n}{1+\|v-w\|_n}\right) < \frac{\varepsilon}{2^{n+1}}$$

and hence

$$(2-\varepsilon)\|v-w\|_n < \varepsilon$$

and hence since $\varepsilon \leqslant 1$:

$$\|v-w\|_n < \varepsilon.$$

Thus if $\varepsilon \leqslant 1$, $B_{<\frac{\varepsilon}{2^{n+1}}}(v) \subset N(v;\|\ \|_n;\varepsilon)$.

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Now it is time to move to examples:

<u>Example V.9</u>: Let X be a locally compact Hausdorff space. For every compact subset $K \subset X$ define

$$\|f\|_K := \sup_{x \in K} |f(x)|, \quad f \in C(X).$$

Then $\{\| \ \|_K : K \subset X, \text{ compact}\}$ is a sufficient family of seminorms. We have that $f_n \to f$ in this topology $\iff$ f_n converges to f uniformly on every compact subset. Observe that if $X = \bigcup_{n \geq 1} K_n$ is a countable union of compact subsets then $\{\| \ \|_{K_n} : n \geq 1\}$ is a countable sufficient family of seminorms inducing the same topology and hence $C(X)$ is metrizable in this case.

It is left as an exercise to show that if X is <u>not</u> compact, there is no norm on $C(X)$ inducing the same topology.

<u>Example V.10.</u> Let $(X, \mathcal{F}, \mu)$ be a triple consisting of a locally compact Hausdorff space X, μ a positive regular Borel measure on X and $\mathcal{F}$ the σ-algebra of μ-measurable sets. Define

$$L^p_{loc}(X) := \{ f : X \to \mathbb{C} : \text{measurable such that } \forall K \subset X \text{ compact}, \; f \cdot \chi_K \in L^p(X) \}.$$

Then $\|f\|_{p,K} := \|f \cdot \chi_K\|_p$, $K \subset X$ compact defines on $L^p_{loc}(X)$ a sufficient family of semi-norms. Observe that if X is a countable union of compact sets, the topology on $L^p_{loc}(X)$ is metrizable.