## Proof of Lemma III.24.

(1) ⇒ (2) Let $\nu = \sum_{i=1}^{n} \lambda_i \delta_{x_i}$, be a convex combination of Dirac measures: $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. Then $\forall f \in C(X)$

$$\int_X \int_X f(x) f(y) K(x,y) \, d\nu(x) \, d\nu(y)$$

$$= \sum_{i,j=1}^{n} f(x_i) f(x_j) \lambda_i \lambda_j K(x_i, x_j) \geq 0.$$

Now the set of extreme points of $M^1(X)$ is $\{\delta_x : x \in X\}$ and hence by Krein–Milman there is a sequence $(\nu_k)_{k \geq 1}$ of convex combinations of Dirac measures such that $\nu_k \to \mu$ in weak$^*$-topology.

Using that $\left\{ \sum_{i,j} f_i(x)\, g_j(y) \;:\; \begin{array}{l} f_i \in C(X) \\ g_j \in C(X) \end{array} \right\}$

is dense in $C_b(X \times X)$ we deduce that

$\nu_n \times \nu_n \longrightarrow \mu \times \mu$ in the weak$^*$-topology

of $M^1(X \times X)$. This implies (2).

(2) $\Rightarrow$ (3) follows from the density of

$C_b(X)$ in $L^2(X, \mu)$.

(3) $\Rightarrow$ (1). Fix $c_1, \dots, c_n \in \mathbb{R}$ and

$x_1, \dots, x_n$ in $X$. Let $\varepsilon > 0$ and $U_i \ni x_i$

open s.t. $|K(x,y) - K(x_i, x_j)| < \varepsilon \quad \forall (x,y) \in U_i \times U_j$.

Observe that $\mu(U_i) > 0$ by hypothesis.

Now let $f = \sum c_i \dfrac{\chi_{U_i}}{\mu(U_i)}$.

Then:

$$0 \leq \langle T_K f, f \rangle = \sum_{i,j=1}^{n} c_i c_j \frac{1}{\mu(U_i)\mu(U_j)} \int_{U_i \times U_j} K(x,y)\, d\mu(x)\, d\mu(y)$$

From $|K(x,y) - K(x_i, x_j)| < \varepsilon \quad \forall (x,y) \in U_i \times U_j$

we get:

$$\left| \langle T_K f, f \rangle - \sum_{i,j=1}^{n} c_i c_j K(x_i, x_j) \right| \leq \varepsilon \left( \sum_{i=1}^{n} |c_i| \right)^2 .$$

Since $\varepsilon > 0$ was arbitrary this implies (1).