non-measurable sets.

A deeper fact is that it is not possible to extend Lebesgue measure on $\mathbb{R}^d$, $d \geq 3$, as a finitely-additive measure on all subsets of $\mathbb{R}^d$ so that it both translation and rotation invariant!

Here we are going to treat the case $d=1$ which proceeds in two steps, the first of which contains the main idea ~~and~~ based on the use of Hahn-Banach; the second step, more formal can be read in Stein-Shakarchi, second half of page 26 and first half of page 27.

Let $\mathbb{R}/\mathbb{Z}$ be the group of real numbers mod 1, that is, the quotient of the abelian group $\mathbb{R}$ by the subgroup $\mathbb{Z}$ and $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the canonical projection. Let

$\ell^\infty(\mathbb{R}/\mathbb{Z}) = \{ f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R} : f \text{ is bounded} \}.$

We say that $f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$ is measurable if $f \circ \pi : \mathbb{R} \longrightarrow \mathbb{R}$ is $\lambda$-measurable wrt the Lebesgue measure $\lambda$ normalized so that $\lambda([0,1]) = 1$. For $f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$ measurable we define

$$\int_{\mathbb{R}/\mathbb{Z}} f \, d\lambda := \int_0^1 (f \circ \pi)(x) \, d\lambda(x)$$

which exists since $f \circ \pi$ is bounded and measurable.

Next we have an action of $\mathbb{R}$ by translation on $\ell^\infty(\mathbb{R}/\mathbb{Z})$ defined as follows:

observe that for $x \in \mathbb{R}/\mathbb{Z}$ and $h \in \mathbb{R}$, $x+h \in \mathbb{R}/\mathbb{Z}$ is well defined. Then for $f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$,

$$f_h(x) = f(x+h)$$

<u>Thm II.18</u> There is a linear map

$$I : \ell^\infty(\mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}$$

such that:

(1) $I(f) \geq 0$ if $f \geq 0$

(2) $I(f) = \int_{\mathbb{R}/\mathbb{Z}} f \, d\lambda$ whenever $f$ is measurable

(3) $I(f_h) = I(f) \quad \forall h \in \mathbb{R}, \forall f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$.

<u>Proof</u>. This is going to be an application of Thm II.4 with $V = \ell^\infty(\mathbb{R}/\mathbb{Z})$, $M = \{ f \in \ell^\infty(\mathbb{R}/\mathbb{Z}) : f \text{ measurable} \}$ and the linear form

$$I_0(f) := \int_{\mathbb{R}/\mathbb{Z}} f \, d\lambda \, , \quad f \in M.$$

The key now is to find the appropriate gauge function $p: V \to \mathbb{R}$ such that $I_0(f) \leq p(f)$, $\forall f \in M$.

Banach's ingenious construction goes as follows.

For every pair $(A, \alpha)$ consisting of a finite set $A$ and a function $\alpha: A \to \mathbb{R}$ define

$$M_{(A,\alpha)}(f) := \sup_{x \in \mathbb{R}/\mathbb{Z}} \frac{1}{|A|} \left( \sum_{a \in A} f(x + \alpha(a)) \right), \quad f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$$

where $|A|$ is the cardinality of $A$. Define then,

$$p(f) := \inf \left\{ M_{(A,\alpha)}(f) : A \text{ finite}, \ \alpha: A \to \mathbb{R} \right\}$$

Observe that since

$$-\|f\|_\infty \leqslant M_{(A,\alpha)}(f) \leqslant \|f\|_\infty$$

$p(f)$ is well defined.

To establish that $p$ is a gauge it will be convenient to define for $f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$,

$$S(f) := \sup_{x \in \mathbb{R}/\mathbb{Z}} f(x) \in \mathbb{R}.$$

Then $S$ satisfies the following properties:

(1) $S(c \cdot f) = c\,S(f)$ if $c \geqslant 0$, $f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$.

(2) $S(f_1 + f_2) \leqslant S(f_1) + S(f_2)$, $\forall f_1, f_2 \in \ell^\infty(\mathbb{R}/\mathbb{Z})$

(3) $S(f_h) = S(f)$, $\forall h \in \mathbb{R}$, $\forall f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$.

It is then convenient to rewrite $M_{(A,\alpha)}(f)$ as:

$$S\left(\frac{1}{|A|}\sum_{a \in A} f_{\alpha(a)}\right).$$

From this we deduce

$$(1) \quad M_{(A,\alpha)}(cf) = c\, M_{(A,\alpha)}(f), \quad \forall c \geq 0 \quad \forall f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$$

$$(2) \quad M_{(A,\alpha)}(f_1 + f_2) \leq M_{(A,\alpha)}(f_1) + M_{(A,\alpha)}(f_2) \quad \forall f_1, f_2 \in \ell^\infty(\mathbb{R}/\mathbb{Z})$$

$$(3) \quad M_{(A,\alpha)}(f_h) = M_{(A,\alpha)}(f) \quad \forall h \in \mathbb{R} \quad \forall f \in \ell^\infty(\mathbb{R}/\mathbb{Z}).$$

Property (1) implies immediately that

$$p(c \cdot f) = c\, p(f) \quad \forall c \geq 0 \quad \forall f \in \ell^\infty(\mathbb{R}/\mathbb{Z}).$$

Concerning the second defining property of a gauge we make the following observation: let $(A,\alpha)$, $(B,\beta)$ maps from finite sets to $\mathbb{R}$. ~~Then:~~

Define $\alpha + \beta : A \times B \to \mathbb{R}$, $(a,b) \mapsto \alpha(a) + \beta(b)$.

Then: $M_{A \times B, \alpha+\beta}(g) \le M_{(A,\alpha)}(g)$ (4)

$\forall g \in \ell^\infty(\mathbb{R}/\mathbb{Z})$

$M_{A\times B, \alpha+\beta}(g) \le M_{(B,\beta)}(g)$ (5).

We show (4) as (5) follows from (4) by interchanging the role of $(A,\alpha)$ and $(B,\beta)$. To this end we compute:

$$M_{(A\times B, \alpha+\beta)}(g) = S\left( \frac{1}{|A|\cdot|B|} \sum_{\substack{a\in A \\ b \in B}} g_{\alpha(a)+\beta(b)} \right)$$

$$= S\left( \frac{1}{|B|} \sum_{b\in B} \left( \frac{1}{|A|} \sum_{a\in A} g_{\alpha(a)} \right)_{\beta(b)} \right)$$

$$\underset{((2)\ \text{II}.28)}{\le} \frac{1}{|B|} \sum_{b\in B} S\left[ \left( \frac{1}{|A|} \sum_{a\in A} g_{\alpha(a)} \right)_{\beta(b)} \right]$$

$$\underset{((3)\ \text{II}.28)}{=} \frac{1}{|B|} \sum_{b\in B} S\left( \frac{1}{|A|} \sum_{a\in A} g_{\alpha(a)} \right)$$

$$= M_{(A,\alpha)}(g) \quad \text{which shows (4).}$$

Let now $f_1, f_2 \in \ell^\infty(\mathbb{R}/\mathbb{Z})$, $\varepsilon > 0$, and $(A, \alpha), (B, \beta)$ s.t. $M_{(A,\alpha)}(f_1) < p(f_1) + \varepsilon$

$$M_{(B,\beta)}(f_2) < p(f_2) + \varepsilon.$$

Then:

$$p(f_1 + f_2) \leq M_{(A \times B, \alpha + \beta)}(f_1 + f_2)$$

$$\leq M_{(A \times B, \alpha+\beta)}(f_1) + M_{(A \times B, \alpha+\beta)}(f_2) \quad \left(\text{by } \underset{\text{II}.29}{(2)}\right)$$

$$\leq M_{(A, \alpha)}(f_1) + M_{(B, \beta)}(f_2) \quad \left(\text{by } \underset{\text{II}.30}{(4)}\right)$$

$$< p(f_1) + \varepsilon + p(f_2) + \varepsilon$$

which implies $p(f_1 + f_2) \leq p(f_1) + p(f_2)$

and shows that $p$ is a gauge.

Next we observe that $\forall h \in \mathbb{R}$ and $f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$ measurable, we have:

$$I_0(f) = \frac{1}{|A|} \int_{\mathbb{R}/\mathbb{Z}} \Big( \sum_{a\in A} f_{\alpha(a)} \Big) d\lambda$$

$$\leq \int_{\mathbb{R}/\mathbb{Z}} S \Big( \frac{1}{|A|} \sum_{a\in A} f_{\alpha(a)} \Big) d\lambda$$

$$= M_{(A,\alpha)}(f)$$

which by taking infimum over all $(A,\alpha)$'s implies $I_0(f) \leq p(f) \quad \forall f \in M$.

Let $I : \ell^\infty(\mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}$ be the linear form extending $I_0$ and satisfying

$$I(f) \leq p(f) \quad \forall f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$$

given by Thm. II. 4.

Now we show that $I$ satisfies properties (1), (2), (3) of Thm II. 18

Clearly if $f(x) \leq 0 \quad \forall x \in \mathbb{R}/\mathbb{Z}$ then

then $M_{(A,\alpha)}(f) \leq 0$ and hence $p(f) \leq 0$.

Thus $I(f) \leq p(f) \leq 0$. If now $f(x) \geq 0$ $\forall x \in \mathbb{R}/\mathbb{Z}$, then we have $-f(x) \leq 0$ $\forall x \in \mathbb{R}/\mathbb{Z}$, hence:

$$I(-f) \leq 0$$

and by linearity of $I$, $I(f) \geq 0$, which proves (1).

Property (2) is immediate since $I$ extends $I_0$.

For (3) we claim that $p(f - f_h) \leq 0$ $\forall f \in \ell^\infty(\mathbb{R}/\mathbb{Z})$ and $h \in \mathbb{R}$. Indeed, let $N \geq 1$ in $\mathbb{N}$ arbitrary, $A_N = \{1, 2, \dots, N\}$ and $\alpha_N(j) = j \cdot h$. Then the sum entering in the definition of $M_{(A_N, \alpha_N)}(f)$ is:

$$\frac{1}{N} \sum_{j=1}^{N} (f - f_h)(x + j \cdot h)$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left[ f(x + j \cdot h) - f(x + (j+1) \cdot h) \right]$$

$$= \frac{1}{N} \left[ f(x+h) - f(x + (N+1) \cdot h) \right]$$

And hence: $M_{(A_N, \alpha_N)} (f - f_h) \leq \frac{2 \| f \|_\infty}{N} \xrightarrow[N \to \infty]{} 0$

~~and $M_{(A, \alpha)} (f - f_h) \leq 0$~~

which implies $p(f - f_h) \leq 0$.

Thus $I(f - f_h) \leq 0$.

Replacing $f$ by $f_{-h}$ and then $-h$ by $h$

we get $I(f_h - f) \leq 0$

and by linearity $I(f_h) = I(f)$ $\forall h \in \mathbb{R}$

$\forall f \in \mathcal{C}^\infty(\mathbb{R}/\mathbb{Z})$

$\boxed{\Xi}$

For $E \subset \mathbb{R}/\mathbb{Z}$, we say that $E$ is measurable if $\chi_E \in \ell^\infty(\mathbb{R}/\mathbb{Z})$ is and define its Lebesgue measure $\lambda(E) := \int_{\mathbb{R}/\mathbb{Z}} \chi_E \, d\lambda$.

Then we have the following immediate corollary from Thm II. 18:

<u>Corollary II. 19</u>: There is a non-negative set function $\hat{\lambda}$ defined on all subsets of $\mathbb{R}/\mathbb{Z}$ such that

(1) $\hat{\lambda}(E_1 \cup E_2) = \hat{\lambda}(E_1) + \hat{\lambda}(E_2)$

for all disjoint subsets $E_1, E_2$.

(2) $\hat{\lambda}(E) = \lambda(E)$ if $E$ is measurable.

(3) $\hat{\lambda}(E+h) = \hat{\lambda}(E)$ $\forall h \in \mathbb{R}$, $\forall E \subset \mathbb{R}/\mathbb{Z}$.

From this it is not difficult to deduce (see Stein – Shakarchi, pp 26 – 27):

Thm II.20:

There is a function $\hat{\lambda} : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ with the following property:

(1) $\hat{\lambda}(E_1 \cup E_2) = \hat{\lambda}(E_1) + \hat{\lambda}(E_2)$ whenever $E_1, E_2$ are disjoint.

(2) $\hat{\lambda}(E) = \lambda(E)$ whenever $E$ is Lebesgue measurable.

(3) $\hat{\lambda}(E+h) = \hat{\lambda}(E)$ $\forall h \in \mathbb{R}, \forall E \subset \mathbb{R}$.

Corollary II.19 can obviously be refrased in terms of the existance of a finitely additive set function on

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

that is $SO(2)$-invariant and extends Lebesgue (angular) measure on $S^1$.

By contrast for the action of $SO(3)$ on
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$
one has a paradoxical decomposition as was shown by Banach–Tarski.

<u>Thm II.21</u>: There is a countable subset $\Lambda \subset S^2$, a partition
$$S^2 \setminus \Lambda = A_1 \cup A_2 \cup A_3 \cup A_4$$
and two rotations $a, b \in SO(3)$ such that:
$$a(A_2) = A_2 \cup A_3 \cup A_4$$
$$b(A_4) = A_1 \cup A_2 \cup A_4.$$

<u>Corollary II.22</u> There is no $SO(3)$-invariant additive set function on $S^2$ extending Lebesgue measure.

<u>Proof:</u> If $\hat{\lambda} : \mathcal{P}(S^2) \to [0, \infty[$ were such a set function we would first have $\hat{\lambda}(\Lambda) = 0$ since $\Lambda$ is countable.

Then $\hat{\lambda}(A_2) = \hat{\lambda}(aA_2)$

$$= \hat{\lambda}(A_2) + \hat{\lambda}(A_3) + \hat{\lambda}(A_4)$$

which implies $\hat{\lambda}(A_3) = \hat{\lambda}(A_4) = 0$

and similarly,

$$\hat{\lambda}(A_4) = \hat{\lambda}(bA_4) = \hat{\lambda}(A_1) + \hat{\lambda}(A_2) + \hat{\lambda}(A_4)$$

implying $\hat{\lambda}(A_1) = \hat{\lambda}(A_2) = 0$. Thus

$$\hat{\lambda}(S^2) = \underbrace{\hat{\lambda}(\Lambda) + \hat{\lambda}(A_1) + \hat{\lambda}(A_2) + \hat{\lambda}(A_3) + \hat{\lambda}(A_4)}_{=0}$$

which implies $\hat{\lambda}(E) = 0 \quad \forall E \subset S^2$. $\boxtimes$