family of semi-norms. Observe that if X is a countable union of compact sets, the topology on $L^p_{loc}(X)$ is metrizable.

There are many more examples of space of functions on $\mathbb{R}^n$ where the seminorms take into account some local boundedness or local integrability on derivatives.

Let V, W be topological vector spaces defined by families of seminorms $\{\| \|^V_\alpha : \alpha \in A\}$, $\{\| \|^W_\beta : \beta \in B\}$ respectively. Analogous to Thm I. 18 we have a characterization of continuous linear maps $T: V \to W$:

<u>Prop. V.11</u> For a linear map $T : V \to W$ the following are equivalent:

(1) T is continuous

(2) T is continuous at 0

(3) $\forall F \subset B$ finite there exists $G \subset A$ finite s.t. $\forall \beta \in F$:

$$\sup \{ \| T(x) \|_\beta^W : \max_{\alpha \in G} \| x \|_\alpha^V \leq 1 \} < +\infty.$$

<u>Corollary V.12</u> A linear form $f : V \to \mathbb{K}$ is continuous iff there exists $G \subset A$ finite such that:

$$\sup \{ | f(x) | : \max_{\alpha \in G} \| x \|_\alpha^V \leq 1 \} < +\infty.$$

~~In other words $\exists\, C > 0$ and $G \subset A$ finite such that~~

The proofs are left as easy exercises.

Given a topological vector space V (abreviated TVS from now on), we denote by V^* the vector space of continuous linear forms $V \longrightarrow \mathbb{K}$. We have then:

<u>Thm V.13</u> (Hahn–Banach) Let V be a TVS given by a sufficient family $\{\| \ \|_\alpha : \alpha \in A\}$ of seminorms. Then $\forall v \in V, v \neq 0 \ \exists F \in V^*$ with $F(v) \neq 0$.

<u>Proof</u>: Let $v \in V$, $v \neq 0$, and $\alpha \in A$ with $\|v\|_\alpha \neq 0$. We apply Thm II.7 to the ~~gauge~~ seminorm $\| \ \|_\alpha$, the subspace $M = \mathbb{K}\cdot v$, and the linear form $f : \mathbb{K}\cdot v \longrightarrow \mathbb{K}$, $f(\lambda\cdot v) = \lambda\|v\|_\alpha$ to obtain a linear extension $F : V \to \mathbb{K}$

satisfying $|F(w)| \leq \|w\|_\lambda$ $\forall w \in V$. By Cor. V.12 $F \in V^*$ and by construction $F(v) = f(v) = \|v\|_\lambda$. $\boxed{\text{Q.E.D.}}$

V.2. Weak Topologies.

When $(V, \|\cdot\|_V)$ ~~and~~ is a normed space and $(V^*, \|\cdot\|_{V^*})$ its dual we will use the tools from V.1 and use suitable families of seminorms to define new TVS structures on V and V^* that have "fewer" open sets than the corresponding norm topologies. In fact we will apply this construction to any TVS V and its dual V^*, and characterise the resulting topologies as initial topologies.

Def. V.14 Let V be a TVS.

(1) The $\sigma(V, V^*)$-topology on V is the topology defined by the family of seminorms

$$\{ \|\ \|_\lambda : \lambda \in V^* \}$$

where $\|v\|_\lambda := |\lambda(v)|$, $v \in V$. It is often referred to as the weak topology on V.

(2) the $\sigma(V^*, V)$ topology on V^* is the topology defined by the family of seminorms $\{ \| \ \|_v : v \in V \}$ where $\|f\|_v = |f(v)|$, $f \in V^*$. It is often refered to as the weak*-topology on V^*.

Let's observe the following

<u>Lemma V. 15</u>

(1) The family of seminorms defining the $\sigma(V^*, V)$ topology on V^* is sufficient.

(2) If V is a TVS defined by a sufficient family of seminorms then the family of seminorms defining the $\sigma(V, V^*)$ topology on V is sufficient.

Thus the weak* topology on V^* is always Hausdorff, and if V ~~is~~ has a sufficient family of seminorms, the weak topology on V is Hausdorff.

<u>Proof</u>: (1) If $f \in V^*$, $f \neq 0$, then there is $v \in V$ with $f(v) \neq 0$, hence $\|f\|_v \neq 0$.

(2) This follows from Thm V. 13 (Hahn-Banach).

The last assertions follow from Lemma V. 7.

$\square$

~~Here is a useful consequence of the definitions:~~

~~<u>Lemma V. 16</u> Let V be a TVS.~~

~~(1) $\forall f \in V^*$, the linear form $V \to \mathbb{K}$, $v \mapsto f(v)$ is continuous in the weak topology.~~

~~In particular a sequence $(v_n)_{n \geq 1}$ converges~~