VII. Fourier Analysis and Sobolev Embedding Theorems.

Recall that in Example I.12 we defined a family of function spaces on $\mathbb{R}^n$ called Sobolev spaces, denoted $W^{p,k}(\mathbb{R}^n)$ where $p \geq 1$ and $k \in \mathbb{N}$. Loosely speaking $W^{p,k}(\mathbb{R}^n)$ consists of all functions admitting weak derivatives up to order k that are in $L^p(\mathbb{R}^n)$. In this chapter we shall concentrate on $W^{2,k}(\mathbb{R}^n)$ and show that if $k > r + \frac{n}{2}$ this space consists of bounded C^r-functions. The means to achieve this is Fourier Analysis and the Plancherel Theorem to which we now turn.

VII.1. Basic Fourier Analysis on $\mathbb{R}^n$

For a thorough treatment we refer to Iacobelli, Analysis IV, Chap. 3. Here we will recall the basic definitions and theorems strictly necessary to our purposes. Sometimes we include proofs and otherwise refer to Iacobelli's notes.

Let λ be the Lebesgue measure on $\mathbb{R}^n$ and $m = (2\pi)^{-n/2}\lambda$, the normalisation coming from the esthetics of the Fourier inversion theorem. Here $L^p(\mathbb{R}^n)$ will always refer to $L^p(\mathbb{R}^n, m)$. For $x, \xi \in \mathbb{R}^n$ $\langle x, \xi \rangle = \sum_{i=1}^{n} x_i \xi_i$ and we set

$$\langle f, g \rangle = \int f \bar{g} \, dm \quad \text{whenever } |f \cdot g| \in L^1(\mathbb{R}^n).$$

<u>Def. VII.1</u> For $f \in L^1(\mathbb{R}^n)$ define the Fourier transform of f, denoted $\hat{f}$ or $\mathcal{F}(f)$, by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) \, e^{-i\langle x, \xi \rangle} \, dm(x).$$

Recall that

$$C_0(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{C} : f \text{ is continuous and } \lim_{\|\xi\| \to +\infty} f(\xi) = 0 \right\}.$$

With the usual $\| \ \|_\infty$ sup norm this is a Banach space.

<u>Prop. VII. 2</u> If $f \in L^1(\mathbb{R}^n)$ then $\hat{f} \in C_0(\mathbb{R}^n)$ and the operator $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ has norm $\|\mathcal{F}\| \leq 1$.

[Riemann–Lebesgue; Iacobelli Thm 3.3].

Recall $(\lambda(x) f)(y) = f(y-x)$, f any function. The following continuity property of the translation operator is fundamental;

<u>Lemma VII.3</u> Let $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^n)$. Then

$$\mathbb{R} \longrightarrow L^p(\mathbb{R}^n)$$
$$x \longmapsto \lambda(x) f$$

is continuous.

<u>Remark VII. 4.</u> Observe that for $f \in L^\infty(\mathbb{R}^n)$ $x \mapsto \lambda(x) f$ is continuous iff f coincides almost everywhere with a uniformly cont. fct.

Proof: From $\|\lambda(x)f-\lambda(y)f\|_p = \|\lambda(x-y)f-f\|_p$

it suffices to show cont. at $x=0$.

Let first $\varphi \in C_{00}(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subset B_{\leq R}(0)$,

and let $x \in B_{\leq 1}(0)$. Then:

$$\|\lambda(x)\varphi-\varphi\|_p^p = \int_{B_{\leq R+1}(0)} |\varphi(y-x)-\varphi(y)|^p \, dm(y)$$

$$\leq \sup_{y\in B_{\leq R+1}(0)} |\varphi(y-x)-\varphi(y)|^p \, m(B_{\leq R+1}(0))$$

$\longrightarrow 0$ with $x \to 0$ since

φ is uniformly continuous. Now let $f \in L^p(\mathbb{R}^n)$

and $\varepsilon > 0$: since $1 \leq p < +\infty$, let

$\varphi \in C_{00}(\mathbb{R}^n)$ with $\|f-\varphi\|_p < \varepsilon$. Let

$\delta > 0$ with $\|\lambda(x)\varphi-\varphi\|_p < \varepsilon$ $\forall x \in B_{\leq\delta}(0)$.

Then: $\forall x \in B_{\leq\delta}(0)$,

$$\|\lambda(x)f-f\|_p \leq \underbrace{\|\lambda(x)f-\lambda(x)\varphi\|_p}_{\|f-\varphi\|_p} + \|\lambda(x)\varphi-\varphi\|_p + \|\varphi-f\|_p$$

$\leq 3\varepsilon$. $\boxed{S}$

Proof of Prop. VII.2.

We have

$$\hat{f}(\xi+h) - \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\langle x,\xi\rangle}\left[e^{-i\langle x,h\rangle} - 1\right] dm(x).$$

Let $\varepsilon > 0$ and $R > 0$ with

$$\int_{\mathbb{R}^n \setminus B_{\leq R}(0)} |f(x)|\, dm(x) < \varepsilon .$$

Then:

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| \leq \int_{B_{\leq R}(0)} |f(x)| \left| e^{-i\langle x,h\rangle} - 1\right| dm(x)$$

$$+ 2\cdot\varepsilon$$

$$\leq \|f\|_1 \cdot \sup_{x\in B_{\leq R}(0)} \left|e^{-i\langle x,h\rangle} - 1\right| + 2\varepsilon .$$

But for R fixed $e^{-i\langle x,h\rangle} - 1$ conv. uniformly to zero on $B_{\leq R}(0)$ as $h \to 0$.

This shows continuity.

Clearly $|\hat{f}(\xi)| \leq \int |f| dm = \|f\|_1 \quad \forall \xi \in \mathbb{R}^n$.

For the last assertion, which is the Riemann Lebesgue lemma: in the defining integral

$$\hat{f}(\xi) = \int f(x) e^{-i\langle x, \xi\rangle} dm(x)$$

make the change of variable $x \to x - \frac{\pi \xi}{\|\xi\|^2}$ (for $\xi \neq 0$) to get

$$\hat{f}(\xi) = \int \left[\lambda\left(\frac{\pi\xi}{\|\xi\|^2}\right) f\right](x)\, e^{-i\langle x,\xi\rangle} \underbrace{e^{i\pi}}_{-1} dm(x)$$

and add it to the previous identity to get

$$2\hat{f}(\xi) = \int \left(f - \lambda\left(\frac{\pi\xi}{\|\xi\|^2}\right) f\right)(x)\, e^{-i\langle x,\xi\rangle} dm(x)$$

and hence

$$2|\hat{f}(\xi)| \leq \left\| f - \lambda\left(\frac{\pi\xi}{\|\xi\|^2}\right) f \right\|_1 .$$

As $|\xi| \to +\infty$, $\frac{\pi\xi}{\|\xi\|^2} \to 0$; now conclude using Lemma VII. 3. $\boxed{\xi}$

One of the major difficulties with the Fourier transform is that for $f \in L^1(\mathbb{R}^n)$, $\hat{f}$ does not satisfy any global integrability condition on $\mathbb{R}^n$. The next proposition specifies a class of functions in $L^1(\mathbb{R}^n)$ whose Fourier transform is in L^p $\forall p \geq 1$.

Recall that $C^k_{00}(\mathbb{R}^n)$ is the space of compactly supported functions which are continuously differentiable up to order k. In this context let's recall some notations:

Given a multiindex $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$,

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \text{where } |\alpha| = \alpha_1 + \dots + \alpha_n$$

and for $\xi \in \mathbb{R}^n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

<u>Prop. VII.5.</u>

(1) If $f \in C^1_{00}(\mathbb{R}^n)$, then

$$\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = i\,\xi_j \hat{f}(\xi)$$

(2) If $f \in C^k_{00}(\mathbb{R}^n)$, $\forall\ |\alpha| \leq k$:

$$(\widehat{D^\alpha f})(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi)$$

(3) If $f \in C^\infty_{00}(\mathbb{R}^n)$, then

$$\hat{f} \in C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \quad \forall p \geq 1.$$

[Compare with Prop. 3.14, Prop. 3.21 in Iacobelli].

Proof:

(1) $$\left(\widehat{\frac{\partial f}{\partial x_j}}\right)(\xi) = \int \frac{\partial f}{\partial x_j}(x)\, e^{-i\langle x,\xi\rangle}\, dm(x)$$

$$= -\int f(x)\, \frac{\partial\left(e^{-i\langle x,\xi\rangle}\right)}{\partial x_j}\, dm(x)$$

$$= i\xi_j \int f(x)\, e^{-i\langle x,\xi\rangle}\, dm(x)$$

$$= i\xi_j\, \hat{f}(\xi).$$

(2) Follows from (1) by induction.

(3) From (2) we get that $\xi^\alpha \hat{f}(\xi)$ is bounded $\forall \alpha$ and hence so is

$$\prod_j (1+\xi_j^2)\, \hat{f}(\xi). \qquad \boxed{\xi}$$

Assertion (2) is of considerable interest since it shows that $\mathcal{F}$ converts the operator D^α into a simple multiplication.

Let now for $a \in \mathbb{R}$, $a \neq 0$:

$$(S_a f)(x) = f\left(\frac{x}{a}\right).$$

We have then the following properties that are formal verifications:

<u>Lemma VII. 6.</u> For $f \in L^1(\mathbb{R}^n)$:

$$(1)\ \widehat{\left(\lambda(x) f\right)}(\xi) = e^{-i\langle x, \xi\rangle} \hat{f}(\xi)$$

$$(2)\ \left(\lambda(x) \hat{f}\right)(\xi) = \left(\widehat{e^{i\langle x, ?\rangle} f}\right)(\xi)$$

$$(3)\ \left(\widehat{S_a f}\right)(\xi) = \left(a^n S_{1/a} \hat{f}\right)(\xi)$$

$$(4)\ \left(S_a \hat{f}\right)(\xi) = a^n \left(S_{1/a} f\right)^\wedge(\xi).$$

The following computation of Fourier transform is a classical application of complex analysis (see Iacobelli: Example 3.12)

__Example VII.7__ Let $\varphi(x) = e^{-\|x\|^2/2}$.

Then $\hat{\varphi} = \varphi$.

This will actually play a role in the Fourier inversion formula. We now define the inverse Fourier transform:

__Def. VII.8__ For $h \in L^1(\mathbb{R}^n)$ define

$$\tilde{h}(x) = \int_{\mathbb{R}^n} h(\xi)\, e^{i\langle x, \xi\rangle}\, dm(\xi).$$

All properties established for $\hat{f}$ hold for $\tilde{h}$ since $\tilde{h} = \overline{\hat{\bar{h}}}$.

We may also use the notation

$$\mathcal{F}^*(h) = \tilde{h}$$

which is justified by

<u>Lemma $\overline{\text{VII}}$.9</u>

If $f \in L^1(\mathbb{R}^n)$ and $h \in L^1(\mathbb{R}^n)$,

$$\langle \mathcal{F}(f), h \rangle = \langle f, \mathcal{F}^*(h) \rangle.$$

Now we are in position to show a version of the Fourier inversion formula, see Thm 3.25 in Iacobelli.

<u>Theorem $\overline{\text{VII}}$.10.</u> For $f \in C^\infty_{00}(\mathbb{R}^n)$,

$$\mathcal{F}^*\mathcal{F} f = f.$$

<u>Proof.</u> Observe that if $f \in C^\infty_{00}(\mathbb{R}^n)$, by Prop. $\overline{\text{VII}}$.5.(3) $\mathcal{F} f \in L^1(\mathbb{R}^n)$ so $\mathcal{F}^*\mathcal{F} f$ is well defined.

Observe that it suffices to show

$$\mathcal{F}^*\mathcal{F}f(0) = f(0) \quad \forall f \in C_{00}^{\infty}(\mathbb{R}^n).$$

Indeed assuming this, we have

$\forall t \in \mathbb{R}^n$:

$$f(t) = (\lambda(-t)f)(0) = (\mathcal{F}^*\mathcal{F}\,\lambda(-t)f)(0)$$

$$= \mathcal{F}^*\left(e^{i\langle t, ?\rangle}\mathcal{F}f\right)(0) \quad \text{(lemma VII.6 (1))}$$

$$= \mathcal{F}^*(\mathcal{F}f)(t) \quad \text{by definition.}$$

Now we turn to the proof of $\mathcal{F}^*\mathcal{F}f(0) = f(0)$

that is $\quad f(0) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, dm(\xi).$

which can be written as

$$\langle \hat{f}, \mathbb{1} \rangle = f(0).$$

Now let $\varphi \in C^{\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$

such that : $\begin{cases} 0 \leq \varphi \leq 1 , \quad \varphi(0) = 1 \\ \hat{\varphi} \in L^1(\mathbb{R}^n). \end{cases}$

The fact that $\varphi(x) = e^{-\|x\|^2/2}$ fulfills these conditions will be particularly important to us. Recall $(S_a \varphi)(\xi) = \varphi(\xi/a)$. Then, as $a \to +\infty$, $S_a \varphi \to \mathbb{1}$ uniformly on compact subsets of $\mathbb{R}^n$ and $0 \leq S_a \varphi \leq 1$ so that

$$\langle \hat{f}, S_a \varphi \rangle \xrightarrow[a \to \infty]{} \langle \hat{f}, \mathbb{1} \rangle .$$

Now we compute $\langle \hat{f}, S_a \varphi \rangle$ in a different way :

$$\langle \hat{f}, S_a \varphi \rangle = \langle \mathcal{F} f, S_a \varphi \rangle = \langle f, \mathcal{F}^* S_a \varphi \rangle$$

$$= \langle f, a^n S_{1/a} \mathcal{F}^* \varphi \rangle =$$

$$= \int_{\mathbb{R}^n} f(x)\, a^n\, \overline{\mathcal{F}^*\varphi}\,(a.x)\, dm(x)$$

$$= \int_{\mathbb{R}^n} f(x/a)\, \overline{\mathcal{F}^*\varphi}\,(x)\, dm(x).$$

Again, as $a \to +\infty$, $x \mapsto f(x/a)$ converges to $f(0)$ uniformly on compact subsets of $\mathbb{R}^n$; as $|f(x/a)| \leq \|f\|_1$ $\forall a \neq 0$ $\forall x \in \mathbb{R}^n$

and $\mathcal{F}^*\varphi \in L^1(\mathbb{R}^n)$, we get

$$\lim_{a \to +\infty} \int_{\mathbb{R}^n} f(x/a)\, \overline{\mathcal{F}^*\varphi}\,(x)\, dm(x) =$$

$$= f(0) \int_{\mathbb{R}^n} \overline{\mathcal{F}^*\varphi}\,(x)\, dm(x)$$

Finally: $\langle \hat{f}, \mathbb{1} \rangle = f(0) \int_{\mathbb{R}^n} \overline{\mathcal{F}^*\varphi}\,(x)\, dm(x)$

and we conclude by taking

$\varphi(x) = e^{-\|x\|^2/2}$ and using Example $\overline{\underline{\text{VII}}}$.7.

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