III Compact Operators: Spectral Theorem.

The main result of this chapter is the spectral theorem for self-adjoint compact operators on a Hilbert space. In fact many operators arising 'in nature' are compact, examples will arise in the first section of this chapter, while the second is devoted to the proof of the spectral theorem.

III.1. Compact operators and Hilbert–Schmidt operators.

Certain natural classes of operators between Banach spaces have much stronger properties than being bounded.

Def. III.1. A (bounded) operator $T : V \to W$ between Banach spaces is said compact if $\overline{T(B_{\leq 1}(0))}$ is a compact subset of W.

This is visibly equivalent to requiring that $T(B) \subset W$ is compact whenever $B \subset V$ is bounded.

The fundamental example is:

<u>Example III.2</u> If $T : V \to W$ has finite rank then T is compact. <u>Indeed</u> $\mathcal{R}(T)$ is finite dimensional and $T(B_{\leq 1}(0))$ is closed and bounded; it is compact by Heine–Borel.

Let V, W be Banach spaces and $\mathcal{K}(V,W) \subset \mathcal{B}(V,W)$ the subset consisting of compact operators. Then:

<u>Prop. III.3</u>

(1) $\mathcal{K}(V,W)$ is a subvector space of $\mathcal{B}(V,W)$.

(2) If $A \in \mathcal{B}(V,V)$, $T \in \mathcal{K}(V,W)$ and $B \in \mathcal{B}(W,W)$ then $BTA \in \mathcal{K}(V,W)$.

(3) $\mathcal{K}(V,W)$ is closed in $\mathcal{B}(V,W)$ for the operator norm.

The two first assertions are left as (easy) exercises. Concerning the third it will be based on the following compactness criterion in complete metric spaces:

<u>Proposition III.4.</u> Let (X, d) be a complete metric space. For a subset $A \subset X$ the following are equivalent:

(1) the closure $\overline{A}$ of A is compact.

(2) A is totally bounded, that is $\forall \varepsilon > 0 \ \exists F \subset A$ finite such that

$$A \subset \bigcup_{x \in F} B_{\leq \varepsilon}(x) .$$

Proof of Prop. III. 3. (3):

§ We show that $T(B_{\leq 1}(0))$ is totally bounded. Let $\varepsilon > 0$ and n such that $\|T - T_n\| \leq \varepsilon$. For every $x, y \in B_{\leq 1}(0)$ we have then:

$$\|T(x) - T(y)\| \leq \|T(x) - T_n(x)\| + \|T_n(x) - T_n(y)\| + \|T_n(y) - T(y)\|$$

$$\leq 2\|T - T_n\| + \|T_n(x) - T_n(y)\|.$$

Now $T_n(B_{\leq 1}(0))$ is totally bounded, hence $\exists\, F \subset B_{\leq 1}(0)$ finite such that:

$$\|T_n(x) - T_n(y)\| \leq \varepsilon \quad \forall x \in F, \; \forall y \in B_{\leq 1}(0)$$

Which implies

$$\|T(x) - T(y)\| \leq 3 \cdot \varepsilon \quad \forall x \in F, \; \forall y \in B_{\leq 1}(0)$$

and shows that $T(B_{\leq 1}(0))$ is totally bounded. $\square$

<u>Corollary III.5</u> If $T \in \mathcal{B}(V,W)$ is limit of a sequence $(T_n)_{n\geq 1}$ where each T_n has finite rank, then $T \in \mathcal{K}(V,W)$ is compact.

<u>Example III.5</u>[(*)]

Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_i : i \geq 1\}$ and define $T : \bigoplus_{i\geq 1} \mathbb{C} e_i \longrightarrow \bigoplus_{i \geq 1} \mathbb{C} e_i$ by $T(e_i) = \lambda_i e_i$, $\lambda_i \in \mathbb{C}$. Then T extends to a bounded operator $\mathcal{H} \to \mathcal{H}$ iff

$$\sup_{i\geq 1} |\lambda_i| < +\infty .$$

which coincides then with $\|T\|$. We have then that T is compact $\Longleftrightarrow \lim_{i\to\infty} \lambda_i = 0$.

One direction is shown by defining

$$T_n : \mathcal{H} \longrightarrow \mathcal{H}$$

on the basis by $T_n(e_i) = \lambda_i e_i$, $1 \leq i \leq n$

and $T_n(e_j) = 0$ $\quad j \geq n+1$. Then

$$\|T - T_n\| = \sup\{|\lambda_i| : i > n\}$$

and hence $\|T - T_n\| \xrightarrow[n\to\infty]{} 0$.

So for diagonal operators in a Hilbert space as in the above examples, bounded operators correspond to bound sequences, and compact operators to sequences vanishing at infinity. We are going to define a class of operators which in the diagonal case would correspond to the condition $\sum_{i=1}^{\infty} |\lambda_i|^2 < +\infty$

These are the Hilbert – Schmidt operators.

Def III. 6. Let $\mathcal{H}$ be a separable Hilbert space with ONB $\{e_i : i \in \mathbb{N}_{\geq 1}\}$. Then $T \in \mathcal{B}(\mathcal{H})$ is called Hilbert-Schmidt if

$$\sum_{i,j=1}^{\infty} |\langle Te_j, e_i\rangle|^2 < +\infty .$$

Lemma III. 7. If $\{f_i : i \in \mathbb{N}_{\geq 1}\}$ is another orthonormal basis we have

$$\sum_{i,j=1}^{\infty} |\langle Te_j, e_i\rangle|^2 = \sum_{i,j=1}^{\infty} |\langle Tf_j, f_i\rangle|^2$$

Proof:

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\langle Te_j, e_i\rangle|^2 = \sum_{j=1}^{\infty} \|Te_j\|^2$$

$$= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |\langle Te_j, f_i\rangle|^2 = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\langle e_j, T^* f_i\rangle|^2$$

$$= \sum_{i=1}^{\infty} \|T^* f_i\|^2 ,$$ which is independent of

the ONB $\{e_i : i \in \mathbb{N}_{\geq 1}\}$. $\boxed{\xi}$

<u>Def III.8</u> If $T : \mathcal{H} \to \mathcal{H}$ is Hilbert–Schmidt we define its Hilbert–Schmidt norm by

$$\|T\|_2 := \left[\sum_{i,j} |\langle Te_j, e_i\rangle|^2\right]^{1/2}$$

We have then

<u>Corollary III.9</u> If $T : \mathcal{H} \to \mathcal{H}$ is Hilbert–Schmidt, so is T^* and $\|T\|_2 = \|T^*\|_2$.

As one can guess from Example III.5 the operator norm and the Hilbert–Schmidt norm are quite different. However we always have:

<u>Lemma III.10.</u> If $T \in \mathcal{B}(\mathcal{H})$ is Hilbert–Schmidt: $\|T\| \leq \|T\|_2$.

<u>Proof</u>: Let $\{e_i : i \in \mathbb{N}_{\geq 1}\}$ be an ONB of $\mathcal{H}$, $x \in \mathcal{H}$. Then $x = \sum_{i=1}^{\infty} x_i e_i$ with $\sum_{i=1}^{\infty} |x_i|^2 < +\infty$. Therefore,

$$\|T(x)\|^2 = \sum_{i=1}^{\infty} \left| \left\langle T\left(\sum_{j=1}^{\infty} x_j e_j\right), e_i \right\rangle \right|^2$$

$$= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} x_j \langle T(e_j), e_i \rangle \right|^2$$

$$\overset{\text{C.S.}}{\leq} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |x_j|^2 \right) \left(\sum_{j=1}^{\infty} |\langle T(e_j), e_i \rangle|^2 \right)$$

$$= \|x\|^2 \, \|T\|_2^2 .$$

$\boxed{\Xi}$

From this we can conclude:

Proposition III.11 If $T : \mathcal{H} \to \mathcal{H}$ is a Hilbert-Schmidt operator T is compact.

Proof: Let $\{e_i : i \in \mathbb{N}_{\geq 1}\}$ be an ONB. Define $T_n : \mathcal{B}(\mathcal{H})$ by:

$$T_n(e_i) = T(e_i) \qquad 1 \leq i \leq n$$

$$T_n(e_j) = 0 \qquad j \geq n+1.$$

Then T_n has finite rank. In addition $T - T_n$ is Hilbert-Schmidt with

$$\|T - T_n\|_2^2 = \sum_{j=n+1}^{\infty} \|T(e_j)\|^2 \xrightarrow[n\to\infty]{} 0$$

since $\sum_{j=1}^{\infty} \|T(e_j)\|^2 < +\infty$. Since by ~~Cor.~~ lemma III.10 we have $\|T - T_n\| \leq \|T - T_n\|_2$ we conclude $\|T - T_n\| \xrightarrow[n\to\infty]{} 0$ and hence (Prop. III.3.(3))

T is compact. $\boxed{\xi}$

From this we are going to get a large class of concrete compact operators.

Prop. III.12. Let $(X, \mathcal{F}, \mu)$ be a (σ-finite) measure space, $K \in L^2(X \times X, \mu \times \mu)$ and

$$T_K : L^2(X) \to L^2(X)$$

the corresponding bounded operator (see Example I.29). Then T_K is Hilbert-Schmidt, in particular compact and $\|T_K\|_2 = \|K\|_2$.

Proof: We assume that $L^2(X)$ is separable and let $\{e_i : i \in \mathbb{N}_{\geq 1}\}$ be an ONB of $L^2(X)$. Recall that for almost every $x \in K$, $K_x(y) := K(x,y)$ is in $L^2(X)$ we

Compute:

$$\sum_{i=1}^{\infty} \|T_K e_i\|^2 = \sum_{i=1}^{\infty} \int_X |T_K e_i(x)|^2 \, d\mu(x)$$

$$= \sum_{i=1}^{\infty} \int_X |\langle K_x , \bar{e}_i \rangle|^2 \, d\mu(x)$$

$$= \int_X \sum_{i=1}^{\infty} |\langle K_x , \bar{e}_i \rangle|^2 \, d\mu(x)$$

$$= \int_X \sum_{i=1}^{\infty} |\langle \bar{K}_x , e_i \rangle|^2 \, d\mu(x)$$

$$= \int_X \|\bar{K}_x\|_2^2 \, d\mu(x)$$

$$= \|K\|_2 .$$

$\boxed{\wr}$