We now turn to a very useful way of characterizing weak and weak*-topologies by putting them into the larger framework of <u>initial topology</u>, a concept of wide ranging applications.

Let X be a set and $\mathcal{F} = \{(\varphi_i, Y_i) : i \in I\}$ a set of pairs (φ_i, Y_i) where Y_i is a topological space and $\varphi_i : X \to Y_i$ is a map. The question is, find the most "economical" topology on X making all these maps continuous. Of course the discrete topology on X will do, but we want the one with the "least number" of open subsets. Let then $\mathcal{T} \subset \mathcal{P}(X)$ be a topology for which the above maps are continuous. Then $\forall i \in I$, if $U_i \subset Y_i$ is open, we must have $f_i^{-1}(U_i) \in \tau$. Let then

$$\mathcal{O}_1 := \{ f_i^{-1}(U_i) : \underset{\text{open}}{U_i \subset Y_i}, i \in I \}.$$

This is not necessarily a topology as it doesn't necessarily contain all the finite intersection of members of $\mathcal{O}_1$. Let $\mathcal{O}_2$:= all finite intersections of elements in $\mathcal{O}_1$ and let $\tau_{\mathcal{F}}$ be the set of arbitrary unions of elements in $\mathcal{O}_2$. Then $\tau_{\mathcal{F}} \subset \tau$ and the point is:

<u>Lemma V.16</u> $\tau_{\mathcal{F}}$ is a topology.

<u>Proof</u>: That ϕ, X are in $\mathcal{G}_{\mathcal{F}}$, and that $\mathcal{G}_{\mathcal{F}}$ is stable under ~~another~~ arbitrary unions, is clear. For finite intersections, it clearly suffices to show it for two subsets U_1, U_2 in $\mathcal{G}_{\mathcal{F}}$. Let

$$U_1 = \bigcup_{\alpha \in A} V_\alpha \quad , \quad U_2 = \bigcup_{\beta \in B} W_\beta$$

where V_α, W_β are all in $\mathcal{O}_2$. Then

$$V_\alpha = \bigcap_{i \in F_\alpha} \varphi_i^{-1}(\tilde{U}_{i,\alpha}), \quad W_\beta = \bigcap_{j \in G_\beta} \varphi_j^{-1}(\tilde{W}_{j,\beta})$$

where $F_\alpha \subset I$ and $G_\beta \subset I$ are finite and $\tilde{U}_{i,\alpha}$, $\tilde{W}_{j,\beta}$ are open subsets resp. of Y_i, Y_j. Then:

$$U_1 \cap U_2 = \bigcup_{\substack{\alpha \in A \\ \beta \in B}} (V_\alpha \cap W_\beta)$$

and, for each $\alpha \in A$, $\beta \in B$:

$$V_\alpha \cap W_\beta = \bigcap_{i \in F_\alpha} \varphi_i^{-1}(\tilde{U}_{i,\alpha}) \cap \bigcap_{j \in G_\beta} \varphi_j^{-1}(\tilde{W}_{j,\beta})$$

which, since $F_\alpha \cup G_\beta$ is finite, is in $\mathcal{O}_2$, and as a result $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{F}}$.

$\boxed{}$

<u>Definition V.17</u> $\mathcal{T}_{\tilde{\mathcal{F}}}$ is called the <u>initial topology</u> defined by the family $\mathcal{F} = \{(\varphi_i, Y_i)\}_{i \in I}$.

<u>Example V.18</u> Let Y_i, $i \in I$ be a family of topological spaces, $X = \prod_{i \in I} Y_i$ the (set theoretic) cartesian product and

$$pr_i : X \to Y_i$$

the projection onto the "i'th" coordinate.

Then the product topology on X is the

initial topology wrt the family

$$\{(\varphi_i, Y_i) : i \in I\}.$$

Let then $\tau_{\mathcal{F}}$ be the initial topology on X given by a family $\mathcal{F} = \{(\varphi_i, Y_i) : i \in I\}$.

The following two lemmas are quite useful:

<u>Lemma V.19</u> Let Z be a topological space. A map $\psi : Z \to X$ is continuous iff $\varphi_i \circ \psi : Z \to Y_i$ are continuous for all $i \in I$.

<u>Proof</u>: If ψ is continuous then so are all the $\varphi_i \circ \psi : Z \to Y_i$. Conversely if $\varphi_i \circ \psi$ are continuous, then $\forall i$, $\forall U_i \subset Y_i$ open $\psi^{-1}(\varphi_i^{-1}(U_i))$ is open; since $\psi^{-1} : \mathcal{P}(X) \to \mathcal{P}(Z)$ commutes with unions and intersections,

we obtain from the description of $\mathcal{T}_{\mathcal{F}}$ that $\varphi^{-1}(\Omega)$ is open $\forall\, \Omega \in \mathcal{T}_{\mathcal{F}}$. $\boxed{\xi}$

<u>Lemma V.20</u> A sequence $(x_n)_{n\geq 1}$ in X converges to $x \in X$ iff $(\varphi_i(x_n))_{n\geq 1}$ converges to $\varphi_i(x)$ $\forall\, i \in I$.

<u>Proof</u>: left as an exercise. $\boxed{\xi}$

Now let us return to a vector space V over $\mathbb{K}$ and a family of seminorms $\{\| \ \|_\alpha : \alpha \in A\}$. Let $\mathcal{T}$ be the topology on V generated by this family of semi-norms and $\mathcal{T}_{\mathcal{F}}$ the initial topology on V associated to $\mathcal{F} = \{(\| \ \|_\alpha, \mathbb{K}) : \alpha \in A\}$.

<u>Prop. V.21</u> $\mathcal{T} = \mathcal{T}_{\mathcal{F}}$.

Proof: First notice that by construction all the seminorms $\| \cdot \|_\alpha : V \to \mathbb{K}$ are continuous for $\mathcal{T}$; hence $\mathcal{T} \supset \mathcal{T}_{\mathcal{F}}$.

Conversely $N(v, \{\alpha\}, \varepsilon) \in \mathcal{T}_{\mathcal{F}}$ and so is $N(v, F, \varepsilon)$ for every finite $F \subset A$.

Hence $\mathcal{T} \subset \mathcal{T}_{\mathcal{F}}$. $\boxtimes$

We conclude from Prop. V. 21

Cor. V. 22 Let V be a TVS generated by a family of seminorms and V^* its dual.

(1) The $\sigma(V, V^*)$ topology on V is the initial topology for the family

$\mathcal{F} = \{ (f, \mathbb{K}) : f \in V^* \}$.

(2) The $\sigma(V^*, V)$ topology is the initial topology for the family $\{(J(u), \mathbb{K}) : u \in V\}$ where $J(u) : V^* \longrightarrow V$

$f \longmapsto f(u)$.

We deduce from Lemmas V.19 and V.20:

<u>Cor. V.23</u> In the situation of Cor. V.22, let Z be a topological space.

(1) A map $\varphi : Z \to V$ is continuous for the $\sigma(V, V^*)$ topology iff : $f \circ \varphi : Z \to \mathbb{K}$ is continuous $\forall f \in V^*$.

(2) A map $\varphi : Z \to V^*$ is continuous for the $\sigma(V^*, V)$ topology iff $Z \to \mathbb{K}$

$z \mapsto \varphi(z)(u)$

is continuous $\forall u \in V$.

(3) A sequence (v_n) converges in $\sigma(V, V^*)$ to u iff $f(v_n) \longrightarrow f(u)$ $\forall f \in V^*$.

(ii) A sequence (f_n) in V^* converges in $\sigma(V^*, V)$ to $f \in V^*$ iff $f_n(v) \to f(v)$ $\forall v \in V$.

V.3. Normed Spaces and the Banach–Alaoglu Theorem.

Let us now turn to a normed space $(V, \|\cdot\|_V)$ and recall that its dual $(V^*, \|\cdot\|_{V^*})$ is a Banach space. We will often refer to the norm topologies as *strong* topologies; to the $\sigma(V, V^*)$ topology on V as weak topology and the $\sigma(V^*, V)$ topology on V^* as the weak*-topology.

Cor V.23 (1) and (2) has the folling immediate consequence:

<u>Prop. V.24</u> Let $T: V \to W$ be a bounded linear operator of normed spaces $(V, \| \|_V)$, $(W, \| \|_W)$, and $T^*: W^* \to V^*$ its adjoint.

(1) T is continuous for the weak topologies on V and W.

(2) T^* is continuous for the weak* topologies on W^* and V^*.

Not let $(V, \| \|_V)$ be a normed space and $(V^*, \| \|_{V^*})$ its dual. Of course every weak open set is strongly open and the same applies to V^* with its weak*-topology. Now if $F \subset V^*$ is finite and $\varepsilon > 0$

then $N(0; F; r) = \{w \in V : |f(w)| < r \ \forall f \in F\}$

contains the subspace $\bigcap_{f \in F} \operatorname{Ker} f$ which

is of finite codimension in V. Thus if

V is infinite dimensional, the strong

open ball $B_{<\varepsilon}(0)$ is NOT weakly open

and the same observation applies to V^*.

However:

<u>Prop V.25</u> (1) $B^{V}_{\leq r}(0)$ is weak closed.

(2) $B^{V^*}_{\leq r}(0)$ is weak* closed.

<u>Proof:</u>

(1) Recall that $\forall v \in V$,

$$\|v\| = \sup\{|f(v)| : \|f\| \leq 1\}$$

and hence $\|v\| \leq r \iff |f(v)| \leq r \quad \forall f \in B^{V^*}_{\leq 1}(0)$.

Thus $B^V_{\leq r}(0) = \bigcap_{f \in B^{V^*}_{\leq 1}(0)} \{v \in V : |f(v)| \leq r\}$

and hence is weakly closed.

(2) Analogous argument. $\boxed{\xi}$

Examples V.26

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $\{e_n : n \geq 1\}$ an orthonormal basis. Then $\lim_{n \to \infty} e_n = 0$ in the weak topology.

Indeed recall that every $f \in \mathcal{H}^*$ is given by $f(v) = \langle v, x \rangle$ for some $x \in \mathcal{H}$.

We apply the Convergence Criterion of Prop. V.23 (3): let $x = \sum_{n=1}^{\infty} x_n e_n$ with $\sum_{n=1}^{\infty} |x_n|^2 < +\infty$.

Then $\langle e_n, x \rangle = x_n$ and hence $\lim_{n \to \infty} \langle e_n, x \rangle = 0$.

Thus we see that while $B_{\le 1}^{\mathcal{H}}(0)$ is weak closed, the unit sphere

$$S_1^{\mathcal{H}}(0) = \{ u \in \mathcal{H} : \|u\| = 1 \}$$

is not.

<u>Example V. 27</u> (Compare with II. 16).

Let X be a compact Hausdorff space and $C(X, \mathbb{R})$ the Banach space of continuous functions $f : X \to \mathbb{R}$ with the norm $\|f\|_b := \sup_{x \in X} |f(x)|$.

Let $M(X, \mathbb{R})$ be the space of signed regular Borel measures on X, then the $\mathbb{R}$-version of the Riesz representation theorem gives a bijection

$$\mathcal{R} : M(X, \mathbb{R}) \longrightarrow C(X, \mathbb{R})^*$$
$$\mu \longmapsto \Phi_\mu$$

where $\Phi_\mu(f) = \int_X f \, d\mu$, and

$$\|\Phi_\mu\| = |\mu|(X)$$

where $|\mu|$ is the total variation measure of μ. Thus $\mathcal{R}$ is a bijective isometry between the Banach space $(M(X,\mathbb{R}), \| \ \|)$ where $\|\mu\| := |\mu|(X)$ and the Banach space $C(X,\mathbb{R})^*$ dual to $C(X,\mathbb{R})$. The weak*-topology on $C(X,\mathbb{R})^*$ gives via $\mathcal{R}$ a topology on $M(X,\mathbb{R})$ which coincides with the initial topology associated to $\mathcal{F} = \{ (f, \mathbb{R}) : f \in C(X, \mathbb{R}) \}$.

The unit ball in $M(X,\mathbb{R})$ is then given by

$$M_{\leq 1} := \left\{ \mu \in M(X,\mathbb{R}) : \left| \int_X f \, d\mu \right| \leq 1 \quad \forall f \in C(X,\mathbb{R}) \right.$$

$$\left. \text{with } \|f\|_b \leq 1 \right\}.$$

In it there is a particularly interesting subset namely the space of probability measures on X,

$$M^1(X) = \{ \mu : \text{is a positive regular Borel measure on } X \text{ and } \mu(X) = 1 \}$$

$$= \left\{ \mu \in M(X,\mathbb{R}) : \int_X f \, d\mu \geq 0 \right.$$

$$\left. \text{whenever } f \geq 0 \text{ and } \int_X \mathbb{1} \, d\mu = 1 \right\}$$

It is thus a convex weak*-closed subset of the unit ball $M_{\leq 1}$.

<u>Example V. 28</u> Let $X = [0,1]$, $\mathcal{L}$ the Lebesgue measure such that $\mathcal{L}([0,1]) = 1$ and δ_0 the Dirac mass at 0. Then $\delta_0 \in M^1(X)$ and $\mu_n := \frac{1}{n} \mathcal{L} 1_{[0, 1/n]} \in M^1(X)$ $\forall n \geq 1$. For every $f \in C([0,1], \mathbb{R})$ we have

$$\int f \, d\mu_n = \frac{1}{n} \int_0^{1/n} f(x) \, d\mathcal{L}(x) \longrightarrow f(0)$$

by continuity. Thus $\mu_n \to \delta_0$ weak*.

While $\| \mu_n - \delta_0 \| \geq | (\mu_n - \delta_0)(f_n) | = 1$

where f_n is: