III.2. Spectral Theorem for compact self-adjoint operators.

If $\mathcal{H}$ is a $\mathbb{K}$-Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, if $\dim \mathcal{H} < +\infty$ we know that all eigenvalues of T are real and there is an ONB of $\mathcal{H}$ consisting of eigenvectors of T. We are going to generalize this result by replacing the hypothesis $\dim \mathcal{H} < +\infty$ by the hypothesis that T is compact.

For simplicity of notation we will assume that all our Hilbert spaces are $\mathbb{C}$-vector spaces. Analogous results hold over $\mathbb{R}$.

Let V be a Banach space and $T \in \mathcal{B}(V)$.
For $\alpha \in \mathbb{C}$ let $V_\alpha := \{ v \in V : T(v) = \lambda \cdot v \}$
Then V_α is clearly a closed subspace of V.
Recall that α is an eigenvalue of T if $V_\alpha \neq (0)$ in which case the elements of V_α are called the eigenvectors of T corresponding to the eigenvalue α.

<u>Example III.13</u> Let $\mathcal{H} = L^2([0,1], \mathbb{C})$ where we take the usual Lebesgue measure λ on $[0,1]$. Define $Tf(x) = x \cdot f(x)$, $f \in \mathcal{H}$.
Then one verifies that $T^* = T$. However T has no eigenvalue since

$$\alpha \cdot f(x) = x f(x) \quad \lambda\text{-a.e.}$$

$$\Longleftrightarrow \quad (\alpha - x) \cdot f(x) = 0 \quad \lambda\text{-a.e.}$$

and since $\quad \alpha - x \neq 0 \quad \lambda\text{-a.e.}$

we get $\quad f(x) = 0 \quad \lambda\text{-a.e.}$

<u>Theorem III.14</u> (Spectral Theorem) Let $T \in \mathcal{B}(\mathcal{H})$ be compact self-adjoint where $\mathcal{H}$ is a Hilbert space. Then $\mathcal{H}$ has an ONB consisting of eigenvectors. In addition: $\forall \lambda \neq 0$, $\dim \mathcal{H}_\lambda < +\infty$ and

$$\{\lambda \in \mathbb{C} : |\lambda| \geq \varepsilon \text{ and } \dim \mathcal{H}_\lambda > 0\}$$

is finite $\forall \varepsilon > 0$.

The proof is based on two lemmas, one of which are verifications and the second one is trickier.

<u>Lemma III.15</u> Let $T \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space.

(1) If $T = T^*$ and $W \subset \mathcal{H}$ is a T-invariant subspace, so is $W^\perp$.

(2) If $T=T^*$ then $\langle Tv, v\rangle \in \mathbb{R}$ $\forall v \in \mathcal{H}$

in particular all eigenvalues of T are real.

(3) $\|T\| = \sup\left\{ |\langle T(v), w\rangle| : \begin{array}{l}\|v\| \leq 1, \\ \|w\| \leq 1\end{array} \right\}$

(4) If $T=T^*$ and $\lambda \neq \beta$ then ~~E~~ $\mathcal{H}_\lambda$ and ~~E~~ $\mathcal{H}_\beta$ are orthogonal.

<u>Proof</u>: (1) For every $w \in W$ and $v \in W^\perp$ we have

$$\langle w, T(v)\rangle = \langle T(w), v\rangle = 0 \text{ since}$$

$T(w) \in W$, which implies $T(v) \in W^\perp$.

(2) Indeed:

$$\overline{\langle T(v), v\rangle} = \langle v, T(v)\rangle = \langle T^*(v), v\rangle$$

$$= \langle T(v), v\rangle.$$

(3) $\|T\| = \sup\limits_{\|v\| \leq 1} \|T(v)\|$ and

by Cor. II.10, $\|T(v)\| = \sup\limits_{\|w\| \leq 1} |\langle T(v), w\rangle|$.

from which (3) follows.

(4) $\forall\, v \in \mathcal{H}_\lambda$ and $\forall\, w \in \mathcal{H}_\beta$:

$$\lambda \langle v, w \rangle = \langle T(v), w \rangle = \langle v, T(w) \rangle$$

$$= \langle v, \beta w \rangle = \beta \langle v, w \rangle$$

since $\beta \in \mathbb{R}$. And since $\lambda \neq \beta$ this implies $\langle v, w \rangle = 0$ and shows (4).

The next lemma gives the key to the whole theorem:

<u>Lemma III.16</u> Let $T \in \mathcal{B}(\mathcal{H})$ with $T = T^*$. Then:

$$\|T\| = \sup\{\, |\langle T(v), v\rangle| : v \in \mathcal{H} \}.$$

<u>Proof</u>: Let $\alpha = \sup\{\, |\langle T(v), v \rangle| : v \in \mathcal{H} \}$.

Then clearly $\alpha \leq \|T\|$.

We want to show $|\langle T(v), w\rangle| \leq \alpha \|v\| \cdot \|w\|$ which by Lemma III. 15 (3) implies Lemma III.16. As the inequality above is unchanged if we multiply w by an $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, we may assume $\langle T(v), w\rangle \in \mathbb{R}$ and prove the inequality under this hypothesis.

Now: from $T = T^*$ and $\langle T(v), w\rangle \in \mathbb{R}$ we deduce:

$$\langle T(v+w), (v+w)\rangle = \langle T(v), v\rangle + 2\langle T(v), w\rangle + \langle T(w), w\rangle$$

$$\langle T(v-w), (v-w)\rangle = \langle T(v), v\rangle - 2\langle T(v), w\rangle + \langle T(w), w\rangle$$

hence:

$$4\langle T(v), w\rangle = \langle T(v+w), v+w\rangle - \langle T(v-w), v-w\rangle$$

hence $|\langle T(v), w\rangle| \leq \frac{\alpha}{4}\left(\|v+w\|^2 + \|v-w\|^2\right)$

$$\leq \frac{\alpha}{2}\left(\|v\|^2 + \|w\|^2\right).$$

Now $\langle T(v), w\rangle$ is unchanged if we replace v by $\sqrt{a}\cdot v$ and w by $\frac{w}{\sqrt{a}}$, $a > 0$:

$$|\langle T(v), w\rangle| \leq \frac{\alpha}{2}\left(a\|v\|^2 + \frac{1}{a}\|w\|^2\right)$$

We may assume $v \neq 0$ and set $a = \frac{\|w\|}{\|v\|}$ to get $|\langle T(v), w\rangle| \leq \alpha \|v\|\|w\|$. $\boxed{\S}$

Proof of Thm III.14.

(1) Either $\|T\|$ or $-\|T\|$ is an eigenvalue:

We may assume $T \neq 0$; let $(v_n)_{n\geq 1}$ be a sequence with $\|v_n\| = 1$, $\forall n \geq 1$, and $\lim_{n\to\infty} |\langle T(v_n), v_n\rangle| = \|T\|$.

We may assume $\lim_{n\to\infty} \langle T(v_n), v_n\rangle = \lambda$ and proceed to show that λ is an eigenvalue

Clearly $\lambda = \|T\|$ or $\lambda = -\|T\|$.

Since T is compact, modulo passing to a subsequence, we may assume $\lim_{n \to \infty} T(v_n) = w$.

Since $\lim \langle T(v_n), v_n \rangle = \lambda$ and $\lambda \neq 0$ we get that $w \neq 0$. Next we compute,

$$
\begin{aligned}
\|T(v_n) - \lambda v_n\|^2 &= \|T(v_n)\|^2 - 2\lambda \langle T(v_n), v_n \rangle \\
&\quad + \lambda^2 \|v_n\|^2 \\
&\leq 2 \cdot \|T\|^2 - 2\lambda \langle T(v_n), v_n \rangle
\end{aligned}
$$

which with $\lim \langle T(v_n), v_n \rangle = \lambda$ and $\lambda^2 = \|T\|^2$ implies:

$$\lim_{n \to \infty} \|T(v_n) - \lambda v_n\| = 0 .$$

Together with $\lim_{n \to \infty} T(v_n) = w$ this implies $\lim_{n \to \infty} \lambda v_n = w$, that is

$$\lim_{n \to \infty} v_n = \frac{w}{\lambda} \quad \text{and}$$

hence $T(\overset{W}{g}) = \lambda \cdot \overset{W}{g}$.

(2) By Zorn's Lemma we can choose an orthonormal set $\mathcal{A} \subset \mathcal{H}$ of eigenvectors which is maximal among all orthonormal sets of eigenvectors. Let $\langle \mathcal{A} \rangle$ be the $\mathbb{C}$-vector subspace of $\mathcal{H}$ spanned by those vectors and $W := \overline{\langle \mathcal{A} \rangle}$ its closure. We want to show $\overline{W} = \mathcal{H}$. Indeed, otherwise $W^\perp \neq (0)$ and since $T(\overline{W}) \subset \overline{W}$ we have $T(W^\perp) \subset W^\perp$. In addition $W^\perp$ is a Hilbert space and $T|_{W^\perp} : W^\perp \to W^\perp$ is compact. Hence $T|_{W^\perp}$ admits an eigenvector, contradicting the maximality of $\mathcal{A}$.

(3) Let $\varepsilon > 0$ and define:

$$W = \overline{\bigoplus_{|\lambda| \geq \varepsilon} \mathcal{H}_\lambda}$$

Observe that the sum $\bigoplus_{|\lambda| \geq \varepsilon} \mathcal{H}_\lambda$ is direct

since $\forall\, \alpha \neq \beta$, $\mathcal{H}_\alpha \perp \mathcal{H}_\beta$ (Lemma III.15 (4)).

We are going to show that $\dim W < +\infty$.

Since $\mathcal{H}_\lambda \subset \mathcal{H}$ is a closed subspace

$\forall \lambda$, let $P_\lambda : \mathcal{H} \to \mathcal{H}_\lambda$ be the orthogonal projection onto $\mathcal{H}_\lambda$.

Let $v \in B^W_{\leq \varepsilon}(0)$; then $v = \overline{\sum_{|\lambda| \geq \varepsilon}} P_\lambda(v)$

with $\|v\|^2 = \overline{\sum_{|\lambda| \geq \varepsilon}} \|P_\lambda(v)\|^2$.

Define $w := \overline{\sum_{|\lambda| \geq \varepsilon}} \frac{1}{\lambda} P_\lambda(v)$;

which exists since

$$\|w\|^2 = \overline{\sum_{|\lambda| \geq \varepsilon}} \frac{1}{\lambda^2} \|P_\lambda(v)\|^2$$

$$\leq \frac{1}{\varepsilon^2} \sum_{|\lambda| \geq \varepsilon} \| P_\lambda (v) \|^2 = \|v\|^2 \leq 1 .$$

And $T(w) = w$. This shows:

$$T\left(B^W_{\leq 1}(0) \right) \supset B^W_{\leq \varepsilon}(0)$$

which implies that $B^W_{\leq \varepsilon}(0)$ is compact and hence $\dim W < +\infty$.

This implies $\dim \mathcal{H}_\lambda < +\infty \quad \forall \lambda \neq 0$ and the finiteness of

$$\{ \lambda \in \mathbb{C} : |\lambda| \geq \varepsilon , \dim \mathcal{H}_\lambda > 0 \} .$$

$\boxed{\xi}$

<u>Example III.17</u> [Unitary representations of compact groups]. This example is meant to give a glimpse into the field of unitary representations and more specifically in the problem of decomposing them into irreducible ones.

We assume that (X, d) is a compact metric space in which a group G acts by distance preserving bijections:

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g.x \end{aligned}$$

is an action and $d(gx, gy) = d(x, y)$, $\forall g \in G$, $\forall (x,y) \in X \times X$. We assume in addition that G preserves a regular Borel probability measure μ on X.

A fundamental example of such situation is : $X = S^2$, d = angular distance on S^2, $\mathcal{L}$ = Lebesgue measure on S^2 and $G = SO(3)$.

For $g \in G$, $f \in L^2(X, \mu)$ define,

$$\pi(g) f(x) = f(g^{-1} \cdot x).$$

Then, as in Example I.28, $\pi(g)$ is an unitary operator of $L^2(X, \mu)$ and

$$\pi : G \longrightarrow \mathbb{U}(L^2(X, \mu))$$

is a group homomorphism.

Let now $K \in C(X \times X)$ be a continuous kernel such that $K(gx, gy) = K(x, y)$ $\forall g \in G$, $\forall (x, y) \in X \times X$.

Claim : $\pi(g) T_K = T_K \pi(g)$:

Indeed:

$$(\pi(g)\,T_K f)(x) = T_K f(g^{-1}x) = \int_X K(g^{-1}x, y)\, f(y)\, d\mu(y)$$

$$= \int_X K(x, gy)\, f(y)\, d\mu(y) = \int_X K(x,y)\, f(g^{-1}y)\, d\mu(y)$$

$$= (T_K\, \pi(g))\, f(x).$$

This has the following remarkable consequence:

If $K(x,y) = \overline{K(y,x)}$ $\forall (x,y) \in X \times X$ then for every eigenvalue $\lambda \neq 0$ of T_K, the corresponding finite dimensional eigenspace $\mathcal{H}_\lambda \subset L^2(X,\mu)$ is invariant under $\pi(g)$, $g \in G$.

In fact in our situation there is a plethora of such kernels namely: if $k : [0, \infty[\longrightarrow \mathbb{R}$ is continuous, then $K(x,y) := k(d(x,y))$ satisfies:

(1) $T_K^* = T_K$ and T_K is compact.

(2) $\pi(g)\,T_K = T_K\,\pi(g) \quad \forall g \in G.$

This leads then to

Thm III. 18 : $L^2(X,\mu)$ is a direct orthogonal sum of $\pi(G)$-invariant (irreducible) finite dimensional subspaces.

In the case of $SO(3)$ acting on S^2 this decomposition takes the following concrete form. Recall that a polynomial $P \in \mathbb{R}[x,y,z]$ is harmonic if $\Delta P = 0$ where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Let then $\mathcal{H}_n = \left\{ P|_{S^2} : P : \mathbb{R}^3 \to \mathbb{R} \text{ is} \right.$ homogeneous of degree n and harmonic.

Then:

$$L^2(S^2) = \overline{\bigoplus \mathcal{H}_n}$$

and the action of $SO(3)$ on $\mathcal{H}_n$ is irreducible.