Now we come to the central result of this chapter:

<u>Theorem V. 29</u> (Banach–Alaoglu)
Let V be a normed space. Then the unit ball $B^{V^*}_{\leq 1}(0)$ in V^* is weak* compact.

<u>Proof:</u> Define $\forall v \in V$, $D_v := \{ z \in \mathbb{K} : |z| \leq \|v\| \}$. Then D_v is compact $\forall v \in V$, and so is $\prod_{v \in V} D_v$ with product topology. For every $f \in B^{V^*}_{\leq 1}(0)$ and $v \in V$, we have

$$|f(v)| \leq \|v\|$$

and thus
$$I : B^{V^*}_{\leq 1}(0) \longrightarrow \prod_{v \in V} D_v$$
$$f \longmapsto (f(v))_v$$

is well defined.

Since the product topology on $\prod_{v \in V} D_v$ is the initial topology associated to $\{(pr_{D_v}, D_v) : v \in V\}$, it follows from Lemma V.19 that I is continuous, wrt the weak*-topology on $B_{\leq 1}^{V^*}(0)$.

Obviously I is injective.

It is also an open map from $B_{\leq 1}^{V^*}(0)$ to its image $\Delta \subset \prod_{v \in V} D_v$. Indeed $\forall f \in B_{\leq 1}^{V^*}(0)$, $F \subset V$ finite, $\varepsilon > 0$:

$$I(N(f; F; \varepsilon)) = \Delta \cap \{(\xi_v)_{v \in V} : |\xi_v - f(v)| < \varepsilon, \forall v \in F\}.$$

Finally we conclude by showing that Δ is a closed subset of $\prod_{v \in V} D_v$.

Define $\forall x, y \in V$ and $\lambda \in \mathbb{K}$:

$$C_{x,y} := \{ (\xi_\alpha) : \xi_x + \xi_y = \xi_{x+y} \}$$

$$C_{\lambda,x} := \{ (\xi_\alpha) : \xi_{\lambda \cdot x} = \lambda \cdot \xi_x \}$$

Then all these sets are closed and so is Δ ~~since~~

$$\Delta = \Big(\bigcap_{x,y \in V} C_{x,y} \Big) \cap \Big(\bigcap_{\substack{\lambda \in \mathbb{K} \\ x \in V}} C_{\lambda,x} \Big).$$

$\boxed{\xi}$

<u>Remark V.30.</u> Even if V is a Banach space, the closed unit ball $B^V_{\leq 1}(0)$ is not necessarily compact in the weak topology. In fact a Thm of Kakutani asserts that $B^V_{\leq 1}(0)$ is weakly compact if and only if V is reflexive, that is if and only if the linear isometry $J : V \to V^{**}$

from Prop. II.12 is surjective.

One of the important consequences of Banach–Alaoglu in the context of Example V.27 is:

<u>Corollary V.31.</u> Let X be a compact Hausdorff space. Then the space $M^1(X)$ of probability measures on X is weak*-compact.

<u>Proof:</u> It is a weak*-closed subset of the unit ball $M_{\leq 1}$. $\square$

<u>Remark V.32.</u> An equivalent formulation is: the space $M^1(X)$ equipped with the initial topology associated to the family $\mathcal{F} = \{ (f, \mathbb{R}) : f \in C(X, \mathbb{R}) \}$ is compact.

We end this chapter with a construction that will bear its fruits in Chap. VI.

Let X be compact Hausdorff and $\psi \in \mathcal{H}omeo(X)$, a homeomorphism of X.

Then ψ gives rise by precomposition by ψ^{-1} to a linear map:

$$\lambda(\psi) : C(X) \longrightarrow C(X)$$
$$f \longmapsto f \circ \psi^{-1}.$$

Then the following two properties are immediate

$$(1)\ \|\lambda(\psi)(f)\|_b = \|f\|_b\ ,\ \forall f \in C(X)$$
$$(2)\ \lambda(\psi_1 \circ \psi_2) = \lambda(\psi_1) \circ \lambda(\psi_2)$$

In particular $\lambda(\psi)$ is a bijective isometry of $C(X)$ with inverse $\lambda(\psi^{-1})$.

Let $\lambda(\psi)^* : C(X)^* \longrightarrow C(X)^*$ be its adjoint.

Then (3) $\lambda(\Psi)^*: C(X)^* \longrightarrow C(X)^*$ is a bijective isometry (exercise) and Prop. V.24(2) implies

$$(4)\ \lambda(\Psi)^*: C(X)^* \longrightarrow C(X)^* \text{ is}$$

weak* - continuous.

Now by (2) and the properties of adjunction we have:

$$\lambda(\Psi_1 \cdot \Psi_2)^* = (\lambda(\Psi_1)\lambda(\Psi_2))^*$$

$$= \lambda(\Psi_2)^* \lambda(\Psi_1)^*$$

which leads us to define:

$$\lambda^*(\Psi) := [\lambda(\Psi)^*]^{-1}.$$

This way we recover:

$$\lambda^*(\Psi_1 \cdot \Psi_2) = \lambda^*(\Psi_1)\lambda^*(\Psi_2).$$

Now coming back to V.27, let's compute $\lambda^*(\Psi)$ under the identification:

$$\begin{aligned} M(X) &\longrightarrow C(X)^* \\ \mu &\longmapsto \Phi_\mu . \end{aligned}$$

— II — ulm :

We have: $\lambda^*(\psi)(\Phi_\mu) = \Phi_\nu$ and

we proceed to compute ν : We have

$$\lambda^*(\psi)(\Phi_\mu)(f) = \underbrace{[\lambda(\psi)^*]^{-1}}_{\lambda(\psi^{-1})^*}(\Phi_\mu)(f)$$

$$= \Phi_\mu(\lambda(\psi^{-1})(f))$$

$$= \Phi_\mu(f \circ \psi)$$

$$= \int_X f \circ \psi(x)\, d\mu(x)$$

And $\Phi_\nu(f) = \int_X f(y)\, d\mu(y)$.

Thus we have the equality

$$\int_X f(y)\, d\nu(y) = \int_X f \circ \psi(x)\, d\mu(x) \quad \forall f \in C(X)$$

Now let for every Borel set $E \subset X$,

$\nu'(E) = \mu(\psi^{-1}(E))$. Then ν' is a

~~a regular~~ signed regular Borel measure denoted $\Psi_* \mu$, and called the pushforward of the measure μ by Ψ. Then clearly,

$$\nu'(E) = \int_X \chi_{\Psi^{-1}(E)}(x)\, d\mu(x) = \int_X \chi_E \circ \Psi(x)\, d\mu(x)$$

which by using step functions and the dominated convergence theorem implies

$$\int_X f(y)\, d\nu'(y) = \int_X f(\Psi(x))\, d\mu(x) \quad \forall f \in C(X)$$

and hence $\nu' = \nu$.

That is : $\boxed{\lambda^*(\Psi)(\phi_\mu) = \phi_{\Psi_*(\mu)}.}$

Finally we observe from this that

$$\lambda^*(\Psi)(M'(X)) = M'(X) \; . \; \forall \Psi \in \mathcal{H}omeo(X).$$

An interesting point is, that this construction can be generalized in the following way: let $\varphi : X \to Y$ be a continuous map of compact Hausdorff spaces and $\mu \in M(X)$. Then

$$(\varphi_* \mu)(E) := \mu(\varphi^{-1}(E)), \quad \underset{\text{Borel}}{E \subset Y}$$

defines an element $\varphi_* \mu \in M(Y)$ and

(1) $\begin{array}{ccc} M(X) & \longrightarrow & M(Y) \\ \mu & \longmapsto & \varphi_* \mu \end{array}$ is weak*-continuous.

(2) $\varphi_*(M^1(X)) \subset M^1(Y)$.

VI. Convexity; the Kakutani–Markov Fixed Point Theorem, and Krein–Milman.

This chapter has three interrelated Themes.

First we will exploit the analytic form of Hahn–Banach to establish separation properties of convex sets in TVS's whose topology is generated by a family of seminorms.

Then we establish a general fixed point theorem (Kakutani–Markov) that has far reaching consequences; for instance it implies that any homeo of a compact Hausdorff space has an invariant probability measure.

Finally we establish a very geometric result on convex compact subsets of a TVS generated by a sufficient family of semi-norms that says that one can recover

it from its subset of extreme points.
Applied to a homeo of a compact Hausdorff space it implies the existence of an ergodic invariant probability measure.
Unless otherwise specified, all vector spaces in this chapter are over $\mathbb{R}$.

VI.1. Convexity.

Let E be a $\mathbb{R}$-vector space.

Def. VI.1 A subset $A \subset E$ is convex if $\forall v, w \in A$, $tv + (1-t)w \in A$ $\forall t \in [0,1]$.

Example VI.2.

Let $p : E \to \mathbb{R}$ be a gauge. Recall that this means: (1) $p(\lambda \cdot v) = \lambda p(v)$, $\lambda > 0$, $v \in E$.

(2) $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ $\forall v_1, v_2 \in E$.

Then $\forall r \in \mathbb{R}$,

$$P_{<r} := \{ v \in E : p(v) < r \} \text{ is convex.}$$

Indeed $v_1, v_2 \in P_{<r}$ and $t \in (0,1)$

implies:

$$p(tv_1 + (1-t)v_2) \leq p(tv_1) + p((1-t)v_2)$$
$$= t\,p(v_1) + (1-t)\,p(v_2)$$
$$< t\cdot r + (1-t)\cdot r = r.$$

It extends clearly to $t=1$ and $t=0$.

The same argument shows that if one replace $<$ in the definition of $P_{<r}$ by $\leq$ the corresponding subset is convex as well.

This process can be reversed for convex subsets with additional properties.

<u>Def VI.3</u> A subset $A \subset E$ is <u>absorbant</u> if $\forall v \in E$ $\exists \alpha > 0$ such that:

$$\lambda \cdot A \ni v \quad \forall |\lambda| \geq \alpha.$$