and we conclude by taking

$\varphi(x) = e^{-\|x\|^2/2}$ and using Example $\overline{\underline{\text{VII}}}.7$.

$\boxed{3}$

From Thm $\overline{\underline{\text{VII}}}.10$ we immediately conclude

<u>Cor. $\overline{\underline{\text{VII}}}.11.$</u> $\langle \mathcal{F} f, \mathcal{F} g \rangle = \langle f, g \rangle$

$\forall f, g \in C_{00}^{\infty}(\mathbb{R}^n).$

In particular $\|\mathcal{F} f\|_2 = \|f\|_2 \quad \forall f \in C_{00}^{\infty}(\mathbb{R}^n)$

In fact we have the stronger result:

<u>Thm $\overline{\underline{\text{VII}}}.12$</u> (Plancherel) The map

$$\mathcal{F} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$$

extends uniquely to a unitary operator

$$\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

We will need the following lemma left as an exercise:

<u>Lemma VII.13</u> Let B_1, B_2 be Banach spaces with resp. norms $\| \ \|_1, \| \ \|_2$, $D \subset B_1$ a vector subspace and $T : D \to B_2$ a bounded linear operator. Then T extends uniquely to a bounded linear operator

$$T_{ext} : \overline{D} \to B_2 .$$

If in addition $\|T(v)\|_2 = \|v\|_1 \ \forall v \in D$ then the same holds for T_{ext} and $v \in \overline{D}$.

~~With this we can prove most of Plancherel's theorem:~~

~~Proof: Since $\|Ff\|_2 = \|f\|_2 \ \forall f \in C^\infty_{00}(\mathbb{R}^n)$ and $C^\infty_{00}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$~~

<u>Proof of Plancherel</u>: Since $\|\mathcal{F}f\|_2 = \|f\|_2$ $\forall f \in C^\infty_{00}(\mathbb{R}^n)$ and $C^\infty_{00}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, by lemma VII. 13, $\mathcal{F}$ extends uniquely to an isometry $\mathcal{F}_{ext}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

We claim that for $f \in L^1 \cap L^2$, $\mathcal{F}_{ext} f = \hat{f}$.

Indeed by lemma VII. 20, let φ_k, $k \geq 1$ be a sequence in $C^\infty_{00}(\mathbb{R}^n)$ with $\varphi_k \to f$ in L^1 and L^2. Then, $\mathcal{F}_{ext}(f) = \lim_{k\to\infty} \mathcal{F}(\varphi_k)$ in L^2 and by Prop. VII. 2, $\mathcal{F}(\varphi_k) = \hat{\varphi}_k \longrightarrow \hat{f}$ uniformly. This implies that $\mathcal{F}_{ext}(f) = \hat{f}$.

It remains to show surjectivity of $\mathcal{F}_{ext}$.

Let $h \in C^\infty_{00}(\mathbb{R}^n)$; since $\tilde{h} = \overline{\hat{\bar{h}}}$, lemma VII. 5. (3) implies that $\tilde{h} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

and hence $\mathcal{F}\tilde{h} = h$. Thus $\mathcal{F}_{ext}(L^2(\mathbb{R}^n))$ $\supset C_{oo}^\infty(\mathbb{R}^n)$ and since $C_{oo}^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and $\mathcal{F}_{ext}$ is isometric this shows surjectivity. $\boxed{\S}$

VII.2. Convolution.

Def. VII.14. Given $f_1, f_2 : \mathbb{R}^n \to \mathbb{C}$ measurable and $x \in \mathbb{R}^n$ such that $t \to f_1(x-t)\, f_2(t)$ is in $L^1(\mathbb{R}^n)$ we define

$$f_1 * f_2\,(x) = \int_{\mathbb{R}^n} f_1(x-t)\, f_2(t)\, dm(t).$$

We recall the following result from Analysis IV [See Iacobelli Thm 3.7].

Prop. VII.15 Let $1 \leq p \leq +\infty$. Let $f_1 \in L^1(\mathbb{R}^n)$ and $f_2 \in L^p(\mathbb{R}^n)$ then for almost every $x \in \mathbb{R}^n$, $t \to f_1(x-t)\, f_2(t)$ is in $L^1(\mathbb{R}^n)$ and $\|f_1 * f_2\|_p \leq \|f_1\|_1 \|f_2\|_p$.

If $f_1 \in L^1(\mathbb{R}^n)$ and $f_2 \in L^\infty(\mathbb{R}^n)$, then $\forall x \in \mathbb{R}^n$, $t \mapsto f_1(x-t) f_2(t)$ is in $L^1(\mathbb{R}^n)$ and $f_1 * f_2 \in C_b(\mathbb{R}^n)$.

One of the main points of the convolution product is that Fourier transformation transforms it into pointwise product [See Iacobelli, Prop. 3.8]. Here we will use it to construct a sequence $\delta_\varepsilon \in C^\infty_{00}(\mathbb{R}^n)$ such that for $1 \leq p < +\infty$ and every $f \in L^p(\mathbb{R}^n)$, $\delta_\varepsilon * f \to f$ in L^p as $\varepsilon \to 0$ and $\delta_\varepsilon * f \in C^\infty(\mathbb{R}^n)$: such a sequence δ_ε is called approximate identity and ~~cons~~ is a very useful tool.

First we show

<u>Prop. VII.16</u> If $f_1 \in C_{00}^{\infty}(\mathbb{R}^n)$ and $f_2 \in L^p(\mathbb{R}^n)$ then $f_1 * f_2 \in C^{\infty}(\mathbb{R}^n)$ and $\frac{\partial}{\partial x_i}(f_1 * f_2) = \frac{\partial f_1}{\partial x_i} * f_2 \quad 1 \leq i \leq n.$

Here and in the sequel we will need the following lemma which is a straightforward verification using Fubini and left to the reader.

<u>Lemma VII.17</u> Let $(Y, \mathcal{F}, \mu)$ be a σ-finite measure space, $D \subset \mathbb{R}^n$ open, $f : D \times Y \to \mathbb{C}$ a measurable function such that (1) $\forall y \in Y$, $x \mapsto f(x,y)$ is in $C^1(D)$

(2) $\forall x \in D$, $\forall\, 1 \leq i \leq n$,

$f(x,\cdot)$ and $\frac{\partial f}{\partial x_i}(x,\cdot)$ are in $L^1(Y,\mu)$.

Define $\quad F(x) = \int_Y f(x,y)\, d\mu(y)$

$$G_i(x) = \int_Y \frac{\partial f}{\partial x_i}(x,y)\, d\mu(y)$$

If $\forall\ 1 \le i \le n$, G_i is continuous in D

then $F \in C^1(D)$ and $\dfrac{\partial F}{\partial x_i} = G_i$.

<u>Proof of Prop. VII.16</u> :

$$f_1 * f_2(x) = \int_{\mathbb{R}^n} f_1(x-y)\, f_2(y)\, dm(y).$$

We wish to apply Lemma VII.17 to

$f(x,y) = f_1(x-y)\, f_2(y)$ which would

be straightforward if $f_2 \in L^1(\mathbb{R}^n)$.

In order to reduce to this case fix

any $r > 0$ and take $D = B_{<r}(0)$.

Let $r_1 > 0$ with $\operatorname{supp} f_1 \subset B_{<r_1}(0)$.

Then $\forall x \in D$

$$f_1 * f_2(x) = \int_{\mathbb{R}^n} f_1(x-y)\, \chi_{r+r_1}(y)\, f_2(y)\, dm(y)$$

where χ_{r+r_1} is the characteristic function of $B_{<r+r_1}(0)$. Now consider

$$f : D \times \mathbb{R}^n \longrightarrow \mathbb{C}$$

$$f(x,y) = f_1(x-y)\, \chi_{r+r_1}(y)\, f_2(y),$$ and

observe that now $\chi_{r+r_1} f_2 \in L^1(\mathbb{R}^n)$. Then the first and second conditions of Lemma VII.17 are obviously satisfied. In addition

$$G_i(x) = \int_{\mathbb{R}^n} \frac{\partial f_1}{\partial x_i}(x-y)\, \chi_{r+r_1}(y)\, f_2(y)\, dm(y)$$

$$= \frac{\partial f_1}{\partial x_i} * \left(\chi_{r+r_1} f_2\right)$$

is continuous by Prop. VII. 15, and hence since $\mathcal{D}$ is arbitrary, $f_1 * f_2 \in C^1(\mathbb{R}^n)$ and $\frac{\partial}{\partial x_i}(f_1 * f_2) = \frac{\partial f_1}{\partial x_i} * f_2$. Since $\frac{\partial f_1}{\partial x_i} \in C^\infty_{00}(\mathbb{R}^n)$ a simple recurrence shows that $f_1 * f_2 \in C^\infty(\mathbb{R}^n)$. $\boxed{\Sigma}$

Now we turn to the construction of approximate identity: fix any $\delta \in C^\infty_{00}(\mathbb{R}^n)$, $\delta \geq 0$, with $\int_{\mathbb{R}^n} \delta(y)\, dm(y) = 1$ and for $\varepsilon > 0$ let

$$\delta_\varepsilon(y) = \frac{1}{\varepsilon^n}\delta\left(\frac{y}{\varepsilon}\right).$$

Then $\delta_\varepsilon \in C^\infty_{00}(\mathbb{R}^n)$, $\int \delta_\varepsilon(y)\, dm(y) = 1$ and if $\operatorname{supp} \delta \subset B_{\leq r}(0)$ then:

$$\operatorname{supp} \delta_\varepsilon \subset B_{\leq \varepsilon r}(0).$$

Prop. VII. 18

(1) If $f \in C(\mathbb{R}^n)$ then $\delta_\varepsilon * f \to f$ uniformly on compact sets as $\varepsilon \searrow 0$.

(2) Let $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^n)$. Then $\delta_\varepsilon * f \longrightarrow f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.

Proof:

(1) By a simple change of variables

$$\delta_\varepsilon * f(x) = \int_{\mathbb{R}^n} \delta_\varepsilon(y) f(x-y)\, dm(y)$$

and hence

$$\delta_\varepsilon * f(x) - f(x) = \int_{\mathbb{R}^n} \delta_\varepsilon(y) \big(f(x-y) - f(x)\big)\, dm(y)$$

which leads to

$$|\delta_\varepsilon * f(x) - f(x)| \leq \sup_{y \in B_{\leq \varepsilon r}(0)} |f(x-y) - f(x)|$$

and shows (1).

(2) Let $f \in L^p(\mathbb{R}^n)$, $1 \geq \varepsilon > 0$ and $\varphi \in C_{00}(\mathbb{R}^n)$ with $\|f - \varphi\|_p < \varepsilon$.

Then

$$\|\delta_\varepsilon * f - f\|_p \leq \|\delta_\varepsilon * f - \delta_\varepsilon * \varphi\|_p + \|\delta_\varepsilon * \varphi - \varphi\|_p$$
$$+ \|\varphi - f\|_p .$$

Since

$$\|\delta_\varepsilon * f - \delta_\varepsilon * \varphi\|_p = \|\delta_\varepsilon * (f - \varphi)\|_p$$
$$\leq \|\delta_\varepsilon\|_1 \|f - \varphi\|_p \quad \left(\text{Prop. } \overline{\text{VII}}.15\right)$$
$$= \|f - \varphi\|_p$$

We get

$$\|\delta_\varepsilon * f - f\|_p \leq 2 \cdot \varepsilon + \|\delta_\varepsilon * \varphi - \varphi\|_p .$$

Now $\operatorname{supp}(\delta_\varepsilon * \varphi) \subset B_{\leq r}(0) + \operatorname{supp} \varphi := K$

so

$$\|\delta_\varepsilon * \varphi - \varphi\|_p^p = \int_K |\delta_\varepsilon * \varphi(y) - \varphi(y)|^p \, dm(y)$$
$$\leq \sup_{y \in K} |\delta_\varepsilon * \varphi(y) - \varphi(y)|^p \cdot m(K)$$
$$\longrightarrow 0 \text{ as } \varepsilon \to 0 \text{ by (1).} \quad \blacksquare$$

Here is an important corollary

<u>Cor. VII.13</u> Let $\Omega \subset \mathbb{R}^n$ be an open subset, and $1 \leq p < +\infty$. Then $C^\infty_{00}(\Omega)$ is dense in $L^p(\Omega)$.

<u>Proof:</u> We know that since Ω is locally compact and m is Borel regular that $C_{00}(\Omega)$ is dense in $L^p(\Omega)$.

So let $f \in C_{00}(\Omega)$; since $\operatorname{supp} f$ is compact, $\operatorname{supp} f \subset \Omega$ and Ω is open we can find $\varepsilon_0 > 0$ with $\operatorname{supp} f + B_{\leq \varepsilon_0 r}(0) \subset \Omega$.

Then $\forall \varepsilon < \varepsilon_0$, $\operatorname{supp}(\delta_\varepsilon * f) \subset \Omega$, $\delta_\varepsilon * f \in C^\infty_{00}(\Omega)$ and $\delta_\varepsilon * f \to f$ in $L^p(\Omega)$ as $\varepsilon \to 0$. $\boxed{\Sigma}$

Here is another application that enters in the proof of Plancherel:

Lemma VII.20. Let $1 \leq p_1, p_2 < +\infty$ and $f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Then there is a sequence $(\varphi_k)_{k \geq 1}$ in $C_{00}^{\infty}(\mathbb{R}^n)$ such that $\varphi_k \longrightarrow f$ in L^{p_1} and L^{p_2}.

Proof:

Let $\chi_R = \chi_{B(0)_{<R}}$. Let $\varepsilon > 0$ then there is R with:

$$\| f(1-\chi_R) \|_{p_1} < \varepsilon \quad , \quad \| f(1-\chi_R) \|_{p_2} < \varepsilon .$$

Now $f \cdot \chi_R \in L^{p_1} \cap L^{p_2}$ and has compact support. As a result, together with Prop. VII.16 we conclude $\delta_{1/k} * (f \cdot \chi_R) \in C_{00}^{\infty}(\mathbb{R}^n)$ and since by Prop. VII.18,

$\delta_{1/R} * (f \cdot \chi_R) \longrightarrow f \cdot \chi_R$ in $L^{p_1} \cap L^{p_2}$ we are done. $\boxed{\xi}$