Then $\forall r \in \mathbb{R}$,

$$P_{<r} := \{ v \in E : p(v) < r \} \text{ is convex.}$$

Indeed $v_1, v_2 \in P_{<r}$ and $t \in (0,1)$

implies:

$$\begin{aligned} p(tv_1 + (1-t)v_2) &\leq p(tv_1) + p((1-t)v_2) \\ &= t\,p(v_1) + (1-t)\,p(v_2) \\ &< t \cdot r + (1-t) \cdot r = r. \end{aligned}$$

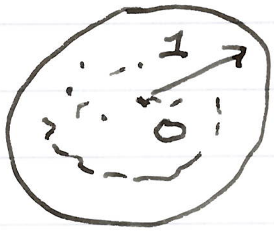
It extends clearly to $t=1$ and $t=0$.

The same argument shows that if one replace $<$ in the definition of $P_{<r}$ by $\leq$ the corresponding subset is convex as well.

This process can be reversed for convex subsets with additional properties.

<u>Def VI.3</u> A subset $A \subset E$ is <u>absorbant</u> if $\forall v \in E$ $\exists \alpha > 0$ such that:

$$\lambda \cdot A \ni v \quad \forall |\lambda| \geq \alpha.$$

Example VI. 4. In $\mathbb{R}^2$ 

is absorbent.

Example VI. 5

Let E be a TVS and $U \ni 0$ open.

Then U is absorbent. Indeed, let $v \in E$:

the map $\mathbb{R} \longrightarrow E$ is continuous,
$t \longmapsto t \cdot v$

in particular at $t = 0$. Hence $\exists\, \varepsilon > 0$

such that $t \cdot v \in U \quad \forall\, |t| \leq \varepsilon$.

Thus $\lambda \cdot U \ni v \quad \forall\, |\lambda| \geq 1/\varepsilon$.

The relationship between convex subsets and

gauges is given by:

Prop. VI. 6. Let E be an $\mathbb{R}$-vector space and

$A \subset E$ such that: (1) A is convex,
(2) $A \ni 0$
(3) A is absorbent.

Then:

$$p_A(v) := \inf\{\rho > 0 : v \in \rho . A\}$$

is a gauge on E. In addition:

(C1) $\{x : p_A(x) < 1\} \subset A \subset \{x : p_A(x) \leq 1\}$

(C2) If $A \subset B$ and B satisfies (1), (2), (3)

then $p_B \leq p_A$.

<u>Proof:</u>

Since A is absorbent, p_A is well defined.

Let $v \in E$ and $\lambda > 0$: then we have

$v \in \rho . A$ for some $\rho > 0 \iff \lambda . v \in \lambda . \rho . A$

and hence $p_A(\lambda . v) = \lambda p_A(v)$.

Let $v, w \in E$ and let ς, λ both > 0

be such that $p_A(v) < \varsigma$, $p_A(w) < \lambda$.

Then $v \in \varsigma . A$, $w \in \lambda . A$, that is

$v = \rho \cdot x$, $w = \lambda \cdot y$ for some x, y in A.

Then:

$$v + w = \rho \cdot x + \lambda \cdot y = (\rho + \lambda) \underbrace{\left[\left(\frac{\rho}{\rho+\lambda}\right) x + \left(\frac{\lambda}{\rho+\lambda}\right) \cdot y \right]}_{\in A, \text{ since } A \text{ is convex}}$$

hence $v + w \in (\rho + \lambda) A$ and hence

$$p_A(v+w) \leq \rho + \lambda .$$

This implies $p_A(v+w) \leq p_A(v) + p_A(w)$.

(C1): If $v \in A$ then $p_A(v) \leq 1$, showing the second inclusion. If $p_A(v) < 1$ then pick $p_A(v) < \lambda < 1$. We have then

$$v \in \lambda \cdot A , \text{ that is } v = \lambda \cdot x , x \in A$$

But since $0 \in A$ and A is convex:

$$v = \lambda \cdot x + (1 - \lambda) \cdot 0 \in A .$$

(C2) We have $A \subset B$: let

$p_A(v) < \rho$, then $v \in \rho \cdot A \subset \rho \cdot B$

which implies $p_B(v) \leq \rho$ and hence

$$p_B(v) \leq p_A(v).$$

$\boxed{\wr}$

Example VI.7

Let V be a TVS defined by a family

$\{\| \ \|_\alpha : \alpha \in A\}$ of seminorms, and

$A = N(0, F, \varepsilon)$ where $\varepsilon > 0$ and $F \subset A$ is

finite. Let's compute p_A: we have for $v \in V$

and $\lambda > 0$,

$$\lambda A \ni v \iff \lambda N(0; F; \varepsilon) \ni v$$
$$\iff \max_{\alpha \in F} \left\| \frac{v}{\lambda} \right\|_\alpha < \varepsilon$$
$$\iff \max_{\alpha \in F} \|v\|_\alpha < \lambda \cdot \varepsilon .$$

and hence $p_A(v) = \frac{1}{\varepsilon} \max_{\alpha \in F} \|v\|_\alpha$

With these tools at hand we can now prove:

Thm VI.8 Let E be ~~an $\mathbb{R}$-vector space~~ a TVS (over $\mathbb{R}$) defined by a family of seminorms $\{\| \ \|_\alpha : \alpha \in A\}$. Let $A \subset E$ be open convex, $\neq \emptyset$ and $x \notin A$. Then there is $F \in E^*$ with

$$F(a) < F(x) \quad \forall a \in A.$$

Proof: Pick $a_0 \in A$; then $A' := A - a_0$ is open, convex, and $A' \ni 0$. Let $p_{A'} : E \to \mathbb{R}$ be the associated gauge by Prop. VI.6.

Define $M := \mathbb{R}(x - a_0)$, $f : M \to \mathbb{R}$

$$\lambda(x - a_0) \to \lambda.$$

Since $x - a_0 \notin A$ we have by Prop. VI.6 (c)

$$p_{A'}(x - a_0) \geq 1.$$

By homogeneity of $p_{A'}$ and linearity of f this implies $f(w) \leq p_{A'}(w)$ $\forall w \in M$.

By Thm II.4 there is $F : E \to \mathbb{R}$ linear extension of f with $F(w) \leq p_{A'}(w)$ $\forall w \in E$.

Let's observe that since A' is open we have $A' = \{ w \in E : p_{A'}(w) < 1 \}$. Indeed the inclusion $\supset$ follows from Prop. VI.6 (c1). For the reverse inclusion let $w \in A'$. Since

$$\begin{aligned} \mathbb{R} &\longrightarrow E \\ t &\longmapsto t \cdot w \end{aligned}$$

is continuous, continuity at $t = 1$ and the fact that A' is open implies there is $\varepsilon > 0$ such that $t \cdot w \in A'$ $\forall t \in [1-\varepsilon, 1+\varepsilon]$. But then $w \in \left(\frac{1}{1+\varepsilon}\right) A'$ hence $p_{A'}(w) < 1$.

Thus $F(a-a_0) \leq p_{A'}(a-a_0) < 1 \quad \forall a \in A$

and $F(v-a_0) = f(v-a_0) = 1$.

$\boxed{\xi}$

With this at hand we can now show that one can separate points from closed convex subsets, more precisely:

<u>Cor. VI.9</u>

Let E be as in Thm VI.8, $A \subset E$ closed convex and $x \notin A$. Then there is $\alpha \in \mathbb{R}$ and $F \in E^*$ such that,

$$F(a) < \alpha < F(x) \quad \forall a \in A.$$

<u>Proof:</u> Since A is closed and $x \notin A$ we can find $U \ni 0$ open with $(x+U) \cap A = \emptyset$.

Let $F \subset A$ finite and $\varepsilon > 0$ such that

$N := N(0; F; \varepsilon) \subset U$. Since $N = -N$, we conclude from $(x + V) \cap A = \phi$ that $x \notin A + N$. Now observe that

$$A + N = \bigcup_{a \in A} (a + N)$$

and is therefore open. It is also convex. Thus by Thm $\overline{\text{VI}}$.8 there is $F \in E^*$ such that $F(a+u) < F(x)$ $\forall a \in A$, $\forall u \in N$.

Since $F \neq 0$, there is $v_0 \in E$ with $F(v_0) \neq 0$ and since N is absorbent (Ex. $\overline{\text{VI}}$ 5) there is $\lambda \neq 0$ with $u_0 := \lambda v_0 \in N$. Thus $F(u_0) \neq 0$ and exchanging u_0 with $-u_0$ if necessary we may assume $F(u_0) > 0$. Thus $F(a) + F(u_0) < F(x)$

With $\alpha := F(x) - F(u_0)$

we get $F(a) < \alpha < F(x) \quad \forall a \in A$.

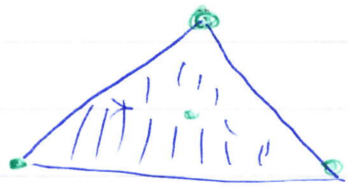
$\boxed{\xi}$

The simple example of A = open disc centered at $0 \in \mathbb{R}^2$ of radius 1 and $x \in \mathbb{R}^2$ with $\|x\| = 1$ shows that the condition in Cor. VI. 9 that A is closed is important.

It is now time to turn to the concept of convex hull of a subset $A \subset E$ in an $\mathbb{R}$-vector space.

<u>Def. VI. 10</u> The convex hull $co(A)$ of a subset $A \subset E$ is the intersection of all convex subsets containing A.

<u>Example $\underline{\text{VI}}$.11</u> :



One shows easily by recurrence that if $v_1, \dots, v_n$ belong to a convex set C then

$$\lambda_1 v_1 + \dots + \lambda_n v_n \in C$$

$\forall (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{\geq 0}$ with $\lambda_1 + \dots + \lambda_n = 1$.

This leads to the following formula for the convex hull of a subset $A \subset E$:

let $\mathbb{R}^{(S)}_{\geq 0} := \{ \lambda : S \to \mathbb{R}_{\geq 0} : \lambda$ has finite support $\}$.

Then

$$co(A) = \left\{ \sum_{a \in A} \lambda(a) \cdot a \;:\; \lambda \in \mathbb{R}^{(A)}_{\geq 0}, \; \sum_{a \in A} \lambda(a) = 1 \right\}.$$

Indeed that $co(A)$ contains the right hand side follows from the above remark while the reversed inclusion follows from the fact that the right hand side is convex, as the following computation shows:

$\forall\, \lambda, \mu \in \mathbb{R}^{(A)}_{\geq 0}$ } and $t \in [0,1]$

$$t\cdot\left(\sum{}' \lambda(a)\, a\right) + (1-t)\left(\sum{}' \mu(a)\, a\right)$$

$$= \sum{}' \big(t\,\lambda(a) + (1-t)\,\mu(a)\big)\, a$$

and $$\sum_a{}' \big(t\lambda(a) + (1-t)\,\mu(a)\big) = t + (1-t) = 1.$$

The following generalizes Prop. V.25

<u>Prop. VI. 12</u> Let $(V, \|\ \|_V)$ be a normed space, and $C \subset V$ convex. Then C is strong closed iff it is weak closed.

Proof :

(1) Weak open ⇒ strong open therefore weak closed ⇒ strong closed.

(2) Assume $G \subset V$ is strong closed; let's show that $V \setminus G$ is weak open.

~~By~~ Let $x_0 \notin G$; by Cor. VI. 9 there is $\alpha \in \mathbb{R}$, $F \in V^*$ with

$$F(c) < \alpha < F(x_0) \quad \forall c \in G.$$

Thus $\{ x \in V : F(x) > \alpha \}$ is a weak open set containing x_0 and disjoint from G.

$\boxed{\$}$

To proceed further we notice

Lemma VI. 13 Let E be a TVS generated by a family $\{ \| \ \|_\alpha : \alpha \in A \}$ of seminorms. Then the closure $\overline{C}$ of a convex subset $C \subset E$ is convex.

Proof: Let $v_1, v_2 \in \overline{C}$, $0 < t < 1$ and $N = N(0; F; \varepsilon) \ni 0$. Then $v_i + N \cap C \neq \phi$; let $u_i \in N$ with $v_i + u_i \in C$. Then

$$[t v_1 + (1-t) v_2] + [t u_1 + (1-t) u_2]$$

$$= t(v_1 + u_1) + (1-t)(v_2 + u_2) \in C .$$ Since N is convex, $t u_1 + (1-t) u_2 \in N$ and hence

$$[t v_1 + (1-t) v_2] + N \cap C \neq \phi. \quad \boxed{}$$

From which we deduce

Prop. VI. 14 Let $(V, \|\ \|_V)$ be a normed space and $C \subset V$ a convex subset. Then its closure $\overline{C}$ for the strong topology coincides with its closure $\overline{C}^w$ for the weak topology.

Proof: Since $\overline{C}^w$ is weakly closed it is strongly closed and since $\overline{C}^w \supset C$ this implies

$\overline{C}^w \supset \overline{C}$. Conversely, by lemma VI.13, $\overline{C}$ is convex; as it is strongly closed it is weakly closed (Prop. VI.12) and hence $\overline{C} \supset \overline{C}^w$. $\boxed{\S}$

Here is a striking corollary:

<u>Cor VI.15</u> Let $(V, \|\cdot\|_V)$ be a normed space and $(v_n)_{n\geq 1}$ a sequence such that

$$v_n \longrightarrow v \quad , \text{ weakly.}$$

Then there is a sequence $w_n \in \mathrm{Co}\{v_n : n\geq 1\}$ such that $w_n \longrightarrow v$, strongly.

<u>Proof</u> By prop. VI.14, $\overline{\mathrm{Co}\{v_n : n\geq 1\}}$ and $\overline{\mathrm{Co}\{v_n : n \geq 1\}}^w$ coincide. $\boxed{\S}$

Another astonishing fact follows from the closed graph theorem and Prop. VI.12:

<u>Prop. VI.16</u>: Let E, F be Banach spaces and $T : E \to F$ a linear map that is continuous for the weak topologies on E and F. Then T is bounded, and conversely.

<u>Proof:</u>

$(\Longleftarrow)$ Prop. V.24. (1).

$(\Longrightarrow)$ $\text{graph}(T) \subset E \times F$ is weak closed, and obviously convex. Hence strong closed which implies T is bounded.

$\boxed{B}$