IV Baire Category and its consequences: Banach – Steinhaus, Open mapping and closed graph theorem.

This chapter is devoted to some of the major theorems in functional analysis. They are all consequences of a result in point-set topology, that is the Baire category theorem. This theorem is a topological analogue of the fact from measure theory that a set of positive measure cannot be countable union of sets of measure zero.

IV . 1 . Baire Category.

The idea of category of a set in a metric space is to describe "smallness" resp. "genericity" in purely topological terms. Its origines lies in the thesis of Baire who answered the following question: given a sequence $f_n : \mathbb{R} \to \mathbb{R}$ of continuous functions converging pointwise to a function $f : \mathbb{R} \to \mathbb{R}$, that is

$$f(x) = \lim_{n\to\infty} f_n(x) \,, \quad \forall x \in \mathbb{R}$$

what can one say about the subset of points in $\mathbb{R}$ at which f is continuous? We will see that set is "big" in a precise way. We now turn to the relevant definitions: let X be a topological space and $S \subset X$ a subset. We recall that the interior S° of S is the union of all open subsets of X contained in S.

<u>Def. IV.1</u> A subset $S \subset X$ is <u>nowhere dense</u> if its closure $\overline{S}$ has empty interior:

$$(\overline{S})^\circ = \emptyset .$$

<u>Examples IV.2</u>

(a) a point in $\mathbb{R}^n$ is nowhere dense $(n \geq 1)$

(b) the Cantor set in $[0,1]$ is (closed and) nowhere dense.

(c) $\mathbb{Q} \subset \mathbb{R}$ is not nowhere dense since $\overline{\mathbb{Q}} = \mathbb{R}$.

However:

(d) $\{(x, 0) : x \in \mathbb{Q}\}$ is nowhere dense in $\mathbb{R}^2$

(e) Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and assume $y \in f(\mathbb{R}^n)$ is a regular value. Then $f^{-1}(y) \subset \mathbb{R}^n$ is nowhere dense in $\mathbb{R}^n$.

<u>Def IV. 3</u>

(1) A set $S \subset X$ is of <u>first category</u> in X if it is a countable union of nowhere dense subsets of X; a subset $S \subset X$ that is not of the first category is of the second category.

(2) A subset $S \subset X$ is <u>generic</u> if its complement is of first category.

<u>Ex IV. 4.</u> $\mathbb{Q}$, while being dense in $\mathbb{R}$, is however of first category and hence $\mathbb{R} \setminus \mathbb{Q}$

, while being dense as well in $\mathbb{R}$, is ~~of second category~~ generic.

The main result of Baire is that $\mathbb{R}$ is of second category in itself. This actually holds for complete metric spaces as the following shows.

<u>Theorem IV.5</u> Let (X, d) be a complete metric space with $X \neq \emptyset$. Then the following assertions hold:

(1) Let $U_j \subset X$, $j \in \mathbb{N}$, be open and dense subsets. Then $U := \bigcap_{j \in \mathbb{N}} U_j$ is dense in X.

(2) Let $F_j \subset X$, $j \in \mathbb{N}$ be a family of closed subsets of X such that

$$\Big(\bigcup_{j \in \mathbb{N}} F_j \Big)^\circ \neq \emptyset$$

then there is $j_0 \in \mathbb{N}$ with $\overset{\circ}{F}_{j_0} \neq \emptyset$.

In particular

(3) Let $X = \bigcup_{j \in \mathbb{N}} F_j$ with F_j closed $\forall j \in \mathbb{N}$. Then $\exists\, j_0 \in \mathbb{N}$ with $\mathring{F}_j \neq \emptyset$.

We begin with

<u>Lemma IV. 6.</u> Let $U \subset X$ and $F := X \setminus U$.
TFAE

(1) U is open and dense in X.

(2) F is closed and nowhere dense in X.

<u>Proof:</u>
(1) $\Rightarrow$ (2): Since U is open, F is closed; if now $V \subset F$ is an open subset of X then $V \cap U = \emptyset$ and since U is dense this implies $V = \emptyset$.

(2) $\Rightarrow$ (1): Since F is closed, U is open; if now $V \subset X$ is any open, non-empty subset of X, since $\mathring{F} = \emptyset$ we have $V \cap U \neq \emptyset$ and U is dense. $\boxed{\xi}$

Proof of Thm IV. 5.

(1) Let $x \in X$ and $r > 0$. We have to show that $B_{<r}(x) \cap U \neq \emptyset$.

Step 1. As U_1 is open and dense,

$$U_1 \cap B_{<r}(x) \text{ is open and } \neq \emptyset.$$

Let $x_1 \in U_1 \cap B_{<r}(x)$ and choose $0 < r_1 < \frac{1}{2}$ such that: $B_{\leq r_1}(x_1) \subset B_{<2r_1}(x_1) \subset U_1 \cap B_{<r}(x)$.

Step 2: Assume we constructed a sequence of points $x_1, \dots, x_n$ and numbers $r_1, \dots, r_n$ such that:

$$B_{\leq r_n}(x_n) \subset B_{r_{n-1}}(x_{n-1}) \subset \cdots \subset B_{\leq r_1}(x_1)$$

$$B_{\leq r_n}(x_n) \subset U_1 \cap \cdots \cap U_n \cap B_{<r}(x)$$

$$0 < r_j < \frac{1}{2^j}$$

Then: $B_{<r_k}(x_k) \cap U_{k+1}$ is non-empty, open.

Let $x_{k+1} \in B_{<r_k}(x_k) \cap U_{k+1}$ and choose

$0 < 2r_{k+1} < \frac{1}{2^k}$ such that:

$$B_{\leq r_{k+1}}(x_{k+1}) \subset B_{<2r_{k+1}}(x_{k+1}) \subset B_{<r_k}(x_k) \cap U_{k+1}.$$

Then $$B_{\leq r_{k+1}}(x_{k+1}) \subset B_{\leq r_k}(x_k)$$

$$B_{\leq r_{k+1}}(x_{k+1}) \subset B_{\leq r_k}(x_k) \cap U_{k+1}$$

$$\subset (U_1 \cap \cdots \cap U_{k+1}) \cap B_{<r}(x).$$

Step 3: We deduce that

$$\bigcap_{k \geq 1} B_{\leq r_k}(x_k) \subset \bigcap_{j \geq 1} U_j \cap B_{<r}(x)$$

$$= U \cap B_{<r}(x).$$

Next, $\forall\, \ell > k \geq 1$ we have

$$d(x_\ell, x_k) \leq \frac{1}{2^k}$$

and hence $(x_n)_{n \geq 1}$ is a Cauchy sequence.

Since X is complete $\lim_{n\to\infty} x_n := y \in X$ exists. But now observe that $\forall n \geq 1$, $(x_k)_{k \geq n}$ is contained in the closed ball ~~$B(x_n)$~~ $B_{\leq r_n}(x_n)$ and hence $y \in B_{\leq r_n}(x_n)$ $\forall n \geq 1$ which implies

$$y \in U \cap B_{<r}(x)$$

and (1) is proven.

Next we show (1) $\Rightarrow$ (3): By contradiction, assume that $X = \bigcup_{j \in \mathbb{N}} F_j$, F_j closed and $\mathring{F}_j = \phi$ $\forall j \in \mathbb{N}$. Then by lemma IV.6, $U_j := X \setminus F_j$ is open and dense. But

$$\bigcap_{j \in \mathbb{N}} U_j = X \setminus \Big(\bigcup_{j \in \mathbb{N}} F_j \Big) = \phi$$

a contradiction.

In order to prove (2) we need

<u>Lemma IV.7</u> Let $\phi \neq U \subset X$ be open in a complete metric space (X, d). Then U satisfies properties (1) and (3) in Thm IV. 5.

<u>Proof:</u> (1) Let U_n, $n \geq 1$ be open dense subsets of U. Observe that U_n is also open in X. The set $V := X \setminus \overline{U}$ is open in X as well. Since U_n is dense in $\overline{U}$ as well, we conclude that $\forall n \geq 1$, $U_n \cup V$ is open and dense in X. By Thm IV.5 (1),

$$\bigcap_{n \geq 1} (U_n \cup V) \text{ is dense in } X.$$

But this intersection coincides with

$$\left(\bigcap_{n \geq 1} U_n \right) \cup V.$$

Let $x \in V$ and $W \ni x$ an open neighborhood of x in $\mathcal{U}$; then W is open in X as well and hence $W \cap \left[\left(\bigcap_{n \geq 1} u_n \right) \cup V \right] \neq \phi$.

But $W \cap V = \phi$ hence

$$W \cap \left(\bigcap_{n \geq 1} u_n \right) \neq \phi.$$

Which shows (2).

(3) Verbatim, same argument as in (1) $\Rightarrow$ (3) in Thm IV.5. $\boxed{S}$

<u>Proof of Thm IV.5. (2):</u>

Let F_j be closed in X and $\mathcal{U} := \left(\bigcup_{j \geq 1} F_j \right)^\circ$ non empty. Then $\mathcal{U} \neq \phi$ and open in X.

Furthermore $F_j \cap \mathcal{U}$ is closed in $\mathcal{U}$, and clearly $\mathcal{U} = \bigcup_{j \geq 1} (F_j \cap \mathcal{U})$.

Hence by lemma IV.7 (3) there is j_0

such that $U \cap F_{j_0}$ contains a non-empty subset W that is open in U hence in X. Thus $\phi \neq W \subset F_{j_0}$ which implies $\mathring{F}_{j_0} \neq \phi$. $\boxtimes$

We can rephrase a consequence of Thm IV.5 as follows:

Corollary IV.8 A complete non-empty metric space is of second category, as is any of its non-empty open subsets.

Corollary IV.9 Any generic subset of a non-empty complete metric space is dense; the same property holds for any non-empty open subset.

Remark IV.10. There is little relation between being generic and being of positive Lebesgue measure, as the following examples show:

(1) $\lambda([0,1]) = 1$ but $[0,1]$ is not generic say in $\mathbb{R}$.

(2) Let
$$\begin{aligned} \mathbb{N} &\longrightarrow \mathbb{Q} \\ k &\longmapsto q_k \end{aligned}$$
be a bijection

and for every $j \geq 1$,
$$U_j := \bigcup_{k \in \mathbb{N}} \left(q_k - 2^{-(j+k+1)}, q_k + 2^{-(j+k+1)} \right)$$

Then $\lambda(U_j) \leq \sum_{k=0}^{\infty} 2^{-(j+k)} = 2^{-(j-1)}$

Clearly U_j is open and dense in $\mathbb{R}$, hence $\bigcap_{j \geq 1} U_j \subseteq \mathbb{R}$ is generic.

But $\lambda\left(\bigcap_{j \geq 1} U_j\right) = \lim_{j \to \infty} \lambda(U_j) = 0$.