I. Banach Spaces, bounded linear Maps: first properties and examples.

I.1. Normed Spaces, Banach Spaces, Examples.

In this course all vector spaces will be over the field $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

These are normed fields where

- for $x \in \mathbb{R}$, $|x| := \max(x, -x)$
- for $z = x + iy \in \mathbb{C}$, $|z| = \sqrt{x^2 + y^2}$.

Let then V be a $\mathbb{K}$-vector space.

<u>Def. I.1</u> A <u>norm</u> on V is a map $V \to \mathbb{R}$, $v \mapsto \|v\|$ satisfying

(1) $\|v\| \geq 0 \quad \forall v \in V$, with equality iff $v = 0$.

(2) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|, \quad \forall v_1, v_2 \in V$.

(3) $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|, \quad \forall \lambda \in \mathbb{K}, \forall v \in V$.

<u>Def. I.2</u>: A normed space is a $\mathbb{K}$-vector space V together with a norm $\| \ \|$; it is often denoted $(V, \| \ \|)$.

Define $\quad d(v_1, v_2) := \|v_1 - v_2\|$. Then properties (1) + (2) are equivalent to the distance axioms. In addition:

$$d(\lambda v_1, \lambda v_2) = |\lambda| \, d(v_1, v_2).$$

Thus a normed space $(V, \| \ \|)$ has a natural distance and in particular is a topological space. By definition of the topology a basis of open sets is given by

$$\{ B_{<r}(x) : x \in V, r > 0 \}$$

where $B_{<r}(x) = \{ y \in V : \|y - x\| < r \}$.

These are called the "open balls"; with these definitions, the sets

$$B_{\leq r}(x) := \{ y \in V : \|y - x\| \leq r \}$$

are closed sets, called "closed balls".

The $\mathbb{K}$-vector space and the topology on V are then compatible in the following sense:

<u>Lemma I.3</u> The maps (1) $\mathbb{K} \times V \to V$

$(\lambda, v) \mapsto \lambda \cdot v$

(2) $V \times V \to V$

$(v_1, v_2) \mapsto v_1 + v_2$

are continuous.

The proof is left as an exercise in Sheet 1.

Later in the course we will have to face more general objects than normed spaces, namely:

<u>Def. I.4</u> A topological vector space is a vector space V endowed with a topology for which the maps (1) + (2) in Lemma I.3 are continuous.

Clearly all concepts pertaining to the theory of metric spaces make sense for normed spaces. The most important one is:

<u>Def. I.5</u> A normed space $(V, \| \, \|)$ is called a <u>Banach</u> space if the underlying

metric space (V, d) is complete.

And:

<u>Def. I.6</u> A normed space $(V, \|\cdot\|)$ is separable if the underlying metric space (V, d) is.

We now turn to examples.

<u>Example I.7</u> Let V be a $\mathbb{K}$-vector space with an inner product

$$\langle \, , \rangle : V \times V \to \mathbb{K}.$$

Then $\|x\| := \sqrt{\langle x, x \rangle}$, $x \in V$, defines a norm on V, (as follows from Cauchy-Schwarz and the def. of inner product).

An inner product space $(V, \langle , \rangle)$ is called a <u>Hilbert space</u> if $(V, \|\,\|)$ is complete, that is a Banach space.

Inner product spaces can be characterized among normed spaces as those whose norm satisfies the parallelogramm law:

$$\|x-y\|^2 + \|x+y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

(see Iacobelli, Analysis IV, 1.1).

<u>Example I.8</u> (see Stein-Shakarchi, Sect. 1.1 and 1.2).

Let $(X, \mathcal{F}, \mu)$ be a <u>measure space</u>. This means that X is a set, $\mathcal{F} \subset \mathcal{P}(X)$ is a σ-algebra of subsets of X (called measurable sets) and $\mu : \mathcal{F} \to [0, +\infty]$ is a σ-additive measure.

For $1 \leq p < +\infty$ let

$L^p(X, \mu, \mathbb{K}) := \{ f : X \longrightarrow \mathbb{K}$ measurable such that $\|f\|_p^p := \int_X |f(x)|^p \, d\mu(x) < +\infty \} / \sim$

where $f_1 \sim f_2$ is $f_1(x) = f_2(x)$ for μ-almost every $x \in X$. Then properties (1) and (3) in the def. of norm are satisfied by $f \longmapsto \|f\|_p$ while the triangle inequality is a consequence of Hölders inequality: if $f, g : X \longrightarrow \mathbb{K}$ are measurable then

$$\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$$ where

$1 \leq p < +\infty$ and q is the <u>conjugate exponent</u>

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We also have an "$\mathcal{L}^p$"-space for $p=\infty$

namely:

$$\mathcal{L}^\infty(X,\mu,\mathbb{K}) := \{ f : X \to \mathbb{K} \text{ measurable}$$

such that $\exists\ 0 < M < +\infty$ with

$$|f(x)| \leq M \text{ for a.e. } x \in X \} / \sim$$

Then for $f \in \mathcal{L}^\infty(X,\mu,\mathbb{K})$,

$$\|f\|_\infty := \inf \{ M > 0 : |f(x)| \leq M \text{ for a.e. } x \in X \}$$

defines a norm on $\mathcal{L}^\infty(X,\mu,\mathbb{K})$.

<u>Special case:</u> $\mathcal{F} = \mathcal{P}(X)$ and $\mu : X \to [0,\infty]$ is the counting measure we denote by $\ell^p(X,\mathbb{K})$ the corresponding $\mathcal{L}^p$-space.

All the L^p-spaces are Banach spaces in the following precise way:

<u>Thm I.9</u> Let $1 \le p \le \infty$ and $(f_n)_{n \ge 1}$ a Cauchy-Sequence in $L^p(X, \mu, \mathbb{K})$. Then there is a subsequence $(f_{n_k})_{k \ge 1}$ converging almost everywhere to a measurable function $f : X \to \mathbb{K}$. In addition, $f \in L^p(X, \mu, \mathbb{K})$ and $\lim_{n \to \infty} \| f - f_n \|_p = 0$.

The case $p = 2$ is special as the norm comes from the innerproduct

$$\langle f, g \rangle := \int_X f(x) \overline{g(x)} \, dx ,$$

$f, g \in L^2$, and $z \mapsto \overline{z}$ on $\mathbb{K}$ denotes ~~con~~ conjugation.

Example I.10

Let X be a topological space and

$C^b(X) := \{ f : X \longrightarrow \mathbb{R} : f$ is continuous and bounded $\}$.

For $f \in C^b(X)$ we let

$$\|f\|_b := \sup_{x \in X} |f(x)|.$$

Then $C^b(X)$ endowed with $\| \ \|_b$ is a Banach space.

Example I.11: Let $\alpha > 0$; then $\Lambda^\alpha(\mathbb{R})$ is the space of all bounded continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$\sup_{t_1 \neq t_2} \frac{|f(t_2) - f(t_1)|}{|t_2 - t_1|^\alpha} < +\infty.$$

Then for $f \in \Lambda^{\alpha}(\mathbb{R})$

$$\|f\|_{\Lambda^{\alpha}} := \sup_{x \in \mathbb{R}} |f(x)| + \sup_{y \neq z} \frac{|f(y)-f(z)|}{|y-z|^{\alpha}}$$

gives rise to a norm on $\Lambda^{\alpha}(\mathbb{R})$ for which it is a Banach space.

We'll see in Exercise Sheet 1 that for $\alpha > 1$ any $f \in \Lambda^{\alpha}(\mathbb{R})$ is constant.

<u>Example I.12</u> The following family of function spaces on $\mathbb{R}^d$, called Sobolev spaces, are fundamental in the study of partial differential equations. First a definition: a function $f \in L^p(\mathbb{R}^d)$ (where the underlying measure is Lebesgue measure on $\mathbb{R}^d$) is said to have weak

derivatives in L^p up to order $k \in \mathbb{N}$ if for every $(\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| := \alpha_1 + \cdots + \alpha_d \leq k$, there is $g_\alpha \in L^p(\mathbb{R}^d)$ with

$$\int_{\mathbb{R}^d} g_\alpha(x)\, \varphi(x)\, d\mathcal{L}(x) = (-1)^{|\alpha|} \int_{\mathbb{R}^d} f(x)\, \partial_x^\alpha \varphi(x)\, d\mathcal{L}(x)$$

$\forall\, \varphi \in C_{00}^\infty(\mathbb{R}^d)$. Observe that if $f \in C^\infty(\mathbb{R}^d)$ then

$$g_\alpha(x) = \partial_x^\alpha f(x) \quad ; \text{ as follows}$$

by integration by parts. In general, if $f \in L^p(\mathbb{R}^d)$ has weak derivatives in L^p up to order k, we write by abuse of terminology,

$$g_\alpha = \partial_x^\alpha f,$$ and denote by $L^p_k(\mathbb{R}^d)$ this function space.

Then: $\|f\|_{L^p_k(\mathbb{R}^d)} := \sum_{|\alpha| \leq k} \|\partial_x^\alpha f\|_p$

turns $L^p_n(\mathbb{R}^d)$ into a Banach space.
A version of the Sobolev embedding theorem says that if $m > \frac{d}{2}$ and $f \in L^2_m(\mathbb{R}^d)$ then f can be corrected on a set of measure zero to become C^k for $k < m - \frac{d}{2}$.

We now turn to properties of finite dimensional normed spaces. The following concept of equivalence for norms will prove useful.

<u>Def. I.13</u>: Two norms $\| \ \|_1$, $\| \ \|_2$ on a vector space V if there is $C > 0$ such that $C^{-1}\|x\|_1 \leq \|x\|_2 \leq C \cdot \|x\|_1$ $\forall x \in V$.

Clearly if $\| \ \|_1$, $\| \ \|_2$ are equivalent norms on V then $(V, \| \ \|_1)$ is a Banach space iff $(V, \| \ \|_2)$ is.

<u>Example I.14</u> Consider $C_{00}(\mathbb{Z}) := \{ f : \mathbb{Z} \to \mathbb{K},\ f \text{ has finite support} \}$ then for $1 \le p_1, p_2 \le \infty$ the norms $\| \ \|_{p_1}$ and $\| \ \|_{p_2}$ on $C_{00}(\mathbb{Z})$ are equivalent iff $p_1 = p_2$.

On finite dimensional spaces we have instead:

<u>Prop. I.15</u> On a finite dimensional $\mathbb{K}$-vector space V all norms are equivalent.

<u>Proof:</u> Let $n = \dim_{\mathbb{K}} V$ and $\Phi : \mathbb{K}^n \to V$ a $\mathbb{K}$-vector space isomorphism. Since any norm on V composed with Φ gives a norm on $\mathbb{K}^n$ it is sufficient to show the prop. for $V = \mathbb{K}^n$. Let $\| \ \|$ be any norm on $\mathbb{K}^n$ and

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} i$$

Then $\forall \; x = \sum_{i=1}^{n} x_i e_i$ we have

$$\|x\| = \Big\| \sum_i x_i e_i \Big\| \leq \sum_i |x_i| \, \|e_i\|$$

$$\leq \|x\|_2 \cdot C_1 \qquad \text{via}$$

where $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$

and $C_1 := \left(\sum_{i=1}^{n} \|e_i\| \right)$

Thus $x \mapsto \|x\|$ is continuous wrt the Euclidean topology on $\mathbb{K}^n$ since

$$\|x - y\| \leq C \cdot \|x - y\|_2 \quad \forall x, y \in \mathbb{K}^n.$$

By Heine-Borel the subset

$$S := \{x \in \mathbb{K}^n : \|x\|_2 = 1\}$$

is compact. Since $x \mapsto \|x\|$ is continuous and non-vanishing on S, there are constants $c_1 > 0$, $c_2 > 0$ with $c_1 \leq \|x\|_2 \leq c_2$, $\forall x \in S$.

Thus $\forall x \in \mathbb{K}^n \setminus \{0\}$

$$c_1 \leq \left\| \frac{x}{\|x\|_2} \right\| \leq c_2$$

which implies that $\| \ \|$ and $\| \ \|_2$ are equivalent. ∎