VII. 3. Weak Derivatives.

Let $\Omega \subset \mathbb{R}^n$ be open.

Recall that for $f \in C^\infty(\Omega)$ and $\varphi \in C^\infty_{00}(\Omega)$

integration by parts gives

$$\int_\Omega f\, D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega (D^\alpha f)\cdot \varphi .$$

We use this to define weak derivatives:

Def. VII.21 Let $f, h \in L^1_{loc}(\Omega)$. Then h is the weak α-th partial derivative of f on Ω if

$$\int f\, D^\alpha \varphi = (-1)^{|\alpha|} \int h\cdot \varphi, \quad \forall \varphi \in C^\infty_{00}(\Omega).$$

Observe that since φ and $D^\alpha \varphi$ are compactly supported, these integrals make sense.

Our first task is to show that if such a weak derivative exists, it is unique. This will follow from

Lemma VII.22. Let $h \in L^1_{loc}(\Omega)$. If $\int h\varphi = 0 \ \forall \varphi \in C^\infty_{00}(\Omega)$ then $h = 0$ a.e.

Proof: Let $x \in \Omega$ arbitrary and let $r > 0$, $\varepsilon > 0$ with $\underset{\leq r+\varepsilon}{B}(x) \subset \Omega$. Let $\chi = \chi_{\underset{\leq r+\varepsilon}{B}(x)}$.

Then $\chi \cdot h \in L^1(\mathbb{R}^n)$. Choose an approximate identity δ_ε where in addition $\operatorname{supp} \delta \subset \underset{<1}{B}(0)$ and $\delta(t) = \delta(-t)$. For every $\varphi \in C^\infty_{00}(\mathbb{R}^n)$ a computation gives

$$\int (\delta_\varepsilon * \chi \cdot h)\varphi = \int \chi \cdot h \cdot (\varphi * \delta_\varepsilon).$$

If now $\operatorname{supp} \varphi \subset \underset{<r}{B}(x)$ then

$$\int (\delta_\varepsilon * \chi \cdot h)\varphi = \int h\,(\varphi * \delta_\varepsilon) = 0$$

since $\operatorname{supp}(\varphi * \delta_\varepsilon) \subset \underset{<r+\varepsilon}{B}(x) \subset \underset{\leq r+\varepsilon}{B}(x) \subset \Omega$.

By Prop VII.16, $\delta_\varepsilon * (\chi \cdot h) \in C^\infty(\mathbb{R}^n)$ and hence its restriction to $B_{<r}(x)$ is L^2. Since $C^\infty_{00}(B_{<r}(x))$ is dense in $L^2(B_{<r}(x))$ we get $\delta_\varepsilon * \chi \cdot h = 0$ in $B_{<r}(x)$. Since $\delta_\varepsilon * \chi \cdot h \longrightarrow \chi \cdot h$ in $L^1(\mathbb{R}^n)$ we get

$$\chi \cdot h = 0 \quad \text{in } B_{<r}(x)$$

and hence $h = 0$ in $B_{<r}(x)$. $\boxed{\xi}$

<u>Notation VII.23</u> If h is the weak α-partial derivative of f, we write $h = D^\alpha_W f$. In particular it follows from Lemma VII.22 that if $f \in C^\infty(\Omega)$, $D^\alpha_W f = D^\alpha f$.

~~Example VII.24. Let $f \in C(\mathbb{R})$ and assume there are $t_1 < t_2 < \dots < t_\ell$ with $f|_{(-\infty, t_1)}$, $f_{(t_1, t_2)}, \dots, f_{(t_{\ell-1}, t_\ell)}, f_{(t_\ell, \infty)}$ are differentiable~~

Example VII .24. Let $\alpha > 0$ and $f(t) = |t|^{\alpha}$

Then $f \in L^1_{loc}(\mathbb{R})$ and the weak derivative

$\frac{d}{dt}_w f$ exists and is $= \alpha \, \mathrm{sgn}(x) \, |x|^{\alpha - 1}$.

Now we can define Sobolev spaces $W^{p,k}(\Omega)$:

Def. VII .25

$W^{p,k}(\Omega) = \{ f : \Omega \to \mathbb{C} : D^{\alpha}_w f$ exists for

all $|\alpha| \leq k$ and $D^{\alpha}_w f \in L^p(\Omega) \}$.

We define the norm on $W^{p,k}(\Omega)$ by :

$$\|f\|_{p,k} := \sum_{|\alpha| \leq k} \|D^{\alpha}_w f\|_p .$$

Prop. VII . 26 $W^{p,k}(\Omega)$ is a Banach space.

Proof: Let $(f_n)_{n \geq 1}$ be a cauchy sequence in $W^{p,k}(\Omega)$. Then $\forall \alpha$, $|\alpha| \leq k$, $(D^{\alpha}_w f_n)_{n \geq 1}$ is a C.S. in $L^p(\Omega)$ and hence has a limit

$f^\alpha \in L^p(\Omega)$; let $f = f^\alpha$ for $\alpha = (0, \dots, 0)$.

Observe that $f^\alpha \in L^p(\Omega) \subset L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$.

By definition we have $\forall\, \alpha$, $|\alpha| \le k$ and

$\varphi \in C^\infty_{00}(\Omega)$:

$$\int f_k D^\alpha \varphi = (-1)^{|\alpha|} \int (D^\alpha_W f_k) \cdot \varphi$$

but as $D^\alpha_W f_k \longrightarrow f^\alpha$ in $L^p(\Omega)$ and

$\varphi \in L^1(\Omega)$ we get

$$\int (D^\alpha_W f_k) \cdot \varphi \longrightarrow \int f^\alpha \cdot \varphi .$$

Since $f_k \longrightarrow f$ in $L^p(\Omega)$ we get

$$\int f D^\alpha \varphi = (-1)^{|\alpha|} \int f^\alpha \cdot \varphi$$

and hence $D^\alpha_W f = f^\alpha$. $\blacksquare$

<u>Remark VII.27</u> $W^{2,k}(\Omega)$ is a Hilbert space; in fact for $f_1, f_2 \in W^{2,k}(\Omega)$

$$\langle f_1, f_2 \rangle = \sum_{|\alpha| \leq k} \langle D_W^\alpha f_1, D_W^\alpha f_2 \rangle$$

leads to the norm

$$\| f \| = \Big(\sum_{|\alpha| \leq k} \| D_W^\alpha f \|_2^2 \Big)^{1/2}$$

which is equivalent to $\| f \|_{2,k}$.

Let $C^\infty_{p,k}(\Omega) := \{ f \in C^\infty(\Omega) :$
$\| D^\alpha f \|_p < +\infty \;\; \forall |\alpha| \leq k \}$.

Then $C^\infty_{p,k}(\Omega) \subset W^{p,k}(\Omega)$ and it is a fact that the former is dense in the latter for any open $\Omega \subset \mathbb{R}^n$. The proof of this is rather delicate and here we will show it for $\Omega = \mathbb{R}^n$.

To this end we collect some simple facts about weak derivatives which will be also useful later on in the proof of the Sobolev embedding theorem.

<u>Lemma VII.28</u>:

(a) If $f \in W^{p,k}(\Omega)$ and $|\alpha|+|\beta| \leq k$ then $D_W^\alpha(D_W^\beta f) = D_W^{\alpha+\beta} f$.

(b) If $f \in W^{p,k}(\Omega)$ and $\varphi \in C_{00}^\infty(\Omega)$ then $\varphi \cdot f \in W^{p,k}(\Omega)$.

(c) If ~~…~~ $\varphi \in C_{00}^\infty(\mathbb{R}^n)$ and $f \in W^{p,k}(\mathbb{R}^n)$ then $\varphi * f \in C_{p,k}(\Omega)$ and

$$D^\alpha(\varphi * f) = \varphi * D_W^\alpha f \quad \forall\, |\alpha| \leq k.$$

Proof:

(a) We have $\forall \varphi \in C_{00}^{\infty}(\Omega)$:

$$\int D_w^{\alpha}(D_w^{\beta} f)\,\varphi = (-1)^{|\alpha|}\int (D_w^{\beta} f)(D^{\alpha}\varphi)$$

$$= (-1)^{|\alpha|+|\beta|}\int f\,(D^{\alpha+\beta}\varphi)$$

$$= \int (D_w^{\alpha+\beta} f)\cdot\varphi\,.$$

By lemma VII.22 we get

$$D_w^{\alpha}(D_w^{\beta} f) = D_w^{\alpha+\beta} f.$$

(b) Let $\psi \in C_{00}^{\infty}(\mathbb{R}^n)$; we may assume $k \geq 1$.

$$\int (\varphi\cdot f)\,\partial_j\psi = \int f\;\varphi\cdot\partial_j\psi = \int f\left[\partial_j(\varphi\cdot\psi) - \psi\,\partial_j\varphi\right]$$

$$= \int f\,\partial_j(\varphi\cdot\psi) - \int f\,\partial_j\varphi\cdot\psi$$

$$= -\int (\partial_j^w f)\,\varphi\cdot\psi - \int f\cdot\partial_j\varphi\cdot\psi$$

$$= - \int \left[(\partial_j^w f) \cdot \varphi + f \cdot \partial_j \varphi \right] \cdot \psi$$

Since $\partial_j^w f \in L^p(\Omega)$ and $f \in L^r(\Omega)$ so is $(\partial_j^w f)\varphi + f \cdot \partial_j \varphi$ and hence $\partial_j^w (\varphi \cdot f)$ exists and is in $L^p(\Omega)$, ~~[illegible]~~ ~~[illegible]~~ and $\partial_j^w (\varphi \cdot f) = \varphi \cdot \partial_j^w f + f \cdot \partial_j \varphi$. $\forall\, 1 \leq j \leq n$. One completes the proof by recurrence on $|\alpha|$ using the formula

$$D^\alpha (\varphi \cdot \psi) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \varphi \, D^{\alpha - \beta} \psi$$

and integration by parts.

(c) We know that $\varphi * f \in C^\infty(\mathbb{R}^n)$ by Prop. VII. 16, and also $\varphi * D_w^\alpha f \in L^p(\mathbb{R}^n)$ $\forall\, |\alpha| \leq k$ by Prop VII. 15. Thus it suffices to show that $D^\alpha(\varphi * f) = \varphi * D_w^\alpha f$

$\forall\, |\alpha| \leq k$. We have $\forall\, \psi \in C_0^\infty(\mathbb{R}^n)$

$$\int (\varphi * D_w^\alpha f)\,\psi = \int (D_w^\alpha f)(\check{\varphi} * \psi)$$

where we set $\check{\varphi}(x) = \varphi(-x)$. The latter equals then

$$= (-1)^{|\alpha|} \int f\, D^\alpha \underbrace{(\check{\varphi} * \psi)}_{\check{\varphi} * D^\alpha \psi}$$

$$= (-1)^{|\alpha|} \int (\varphi * f)\, D^\alpha \psi$$

~~which implies~~ $$= \int D^\alpha(\varphi * f) \cdot \psi$$

and shows $D^\alpha(\varphi * f) = \varphi * D_w^\alpha f$.

Prop. VII.29 $1 \leq p < +\infty$; then $C^\infty_{p,k}(\mathbb{R}^n)$ is dense in $W^{p,k}(\mathbb{R}^n)$.

Proof: Fix an approximation of unity δ ~~with~~ $\cancel{\delta_\varepsilon}$. Let $f \in W^{p,k}(\mathbb{R}^n)$.

By lemma VII.28, $\delta_\varepsilon * f \in C^\infty_{p,k}(\mathbb{R}^n)$

and $D^\alpha(\delta_\varepsilon * f) = \delta_\varepsilon * D^\alpha_W f$, $|\alpha| \leq k$.

By prop. VII.18 (2) we have for $|\alpha| \leq k$,

$$\delta_\varepsilon * D^\alpha_W f \longrightarrow D^\alpha_W f \quad \text{in } L^p(\mathbb{R}^n)$$

and hence $D^\alpha(\delta_\varepsilon * f) \longrightarrow D^\alpha_W f$ in $L^p(\mathbb{R}^n)$.

$\blacksquare$

VII.4. Sobolev embedding theorems.

The aim of this section is to prove

Thm VII.30 (Sobolev) If $f \in W^{2,k}(\mathbb{R}^n)$ and $k > r + \frac{n}{2}$ then $f \in C^r_b(\mathbb{R}^n)$. Moreover the inclusion $W^{2,k}(\mathbb{R}^n) \to C^r_b(\mathbb{R}^n)$ is a bounded operator.

Remark VII.31 More precisely if $f \in W^{2,k}(\mathbb{R}^n)$, f coincides almost everywhere with a C^r-function that is bounded.

Interestingly, while in the statement of the Theorem there is no Fourier transform, the latter is a crucial tool in the proof.

We preceed the proof with three lemmas.

<u>Lemma VII.32</u> If $f \in W^{2,k}(\mathbb{R}^n)$ then

$\forall |\alpha| \leq k \quad \widehat{(D_W^\alpha f)}(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi)$

in particular $\xi^\alpha \hat{f} \in L^2(\mathbb{R}^n) \quad \forall |\alpha| \leq k$.

<u>Proof</u>: As usual the proof is by induction on k and we'll do it for $k = 1$.

Let $\varphi \in C^\infty_{00}(\mathbb{R}^n)$, then since $\partial_j^w f \in L^2(\mathbb{R}^n)$ we have by Plancherel,

$$\begin{aligned} \langle \widehat{\partial_j^w f}, \hat{\varphi} \rangle &= \langle \partial_j^w f, \varphi \rangle \\ &= - \langle f, \partial_j \varphi \rangle \\ &= - \langle \hat{f}, \widehat{\partial_j \varphi} \rangle \quad \text{(Plancherel)} \end{aligned}$$

By Prop. VII.5 (1) we have:

$$\widehat{\partial_j \varphi}(\xi) = i\xi_j \hat{\varphi}(\xi)$$

and thus

$$-\langle \hat{f}, \widehat{\partial_j \varphi} \rangle = -\int \hat{f}(\xi)\, \overline{\widehat{\partial_j \varphi}(\xi)}\, dm(\xi)$$

$$= \int \hat{f}(\xi) i\xi_j \overline{\hat{\varphi}(\xi)}\, dm(\xi)$$

$$= \langle i\xi_j \hat{f}, \hat{\varphi} \rangle$$

And since $\{ \hat{\varphi} : \varphi \in C_{00}^{\infty}(\mathbb{R}^n) \}$ is dense in $L^2(\mathbb{R}^n)$ we get

$$(\widehat{\partial_j^w f})(\xi) = i\xi_j \hat{f}(\xi).$$

∎

For the sake of the applications we have in mind we formulate the next lemma in terms of the inverse Fourier transform, which we recall is given by

$$\tilde{h}(x) = \int_{\mathbb{R}^n} h(\xi)\, e^{i\langle x,\xi\rangle}\, dm(\xi).$$

<u>Lemma VII.33</u> Let $r \in \mathbb{N}$, assume $h \in L^1(\mathbb{R}^n)$ and $\xi^\alpha h \in L^1(\mathbb{R}^n)$ $\forall\, |\alpha| \leq r$. Then $\tilde{h} \in C_b^r(\mathbb{R}^n)$ and

$$D^\alpha \tilde{h}(x) = i^{|\alpha|}\, \widetilde{(\xi^\alpha h)}(x).$$

<u>Proof</u>: As usual, a recurrence on r; we do the case $r = 1$. Now

$$\tilde{h}(x) = \int_{\mathbb{R}^n} h(\xi)\, e^{i\langle x,\xi\rangle}\, dm(\xi)$$

is continuous, and $\forall\ 1 \leq j \leq n$:

$$G_j(x) = \int_{\mathbb{R}^n} h(\xi)\, i\xi_j\, e^{i\langle x,\xi\rangle}\, dm(\xi)$$

is also continuous, since by hypothesis,

$$\xi_j \cdot h \in L^1(\mathbb{R}^n), \quad 1 \leq j \leq n.$$

It follows then from Lemma VII.17 that $\tilde{h} \in C^1(\mathbb{R}^n)$ and $\partial_j \tilde{h} = i\,\widetilde{(\xi_j \cdot h)}$ ☒

<u>Lemma VII.34.</u> Let $r \geq 0$. Assume $f \in L^2(\mathbb{R}^n)$ and $\xi^\alpha \hat{f} \in L^1(\mathbb{R}^n)\ \forall\, |\alpha| \leq r$. Then $f \in C_b^r(\mathbb{R}^n)$ and

$$D^\alpha f = i^{|\alpha|}\,\widetilde{(\xi^\alpha \hat{f})}$$

<u>Proof</u>: Apply the preceeding lemma to

$h = \hat{f}$: then we got that $\tilde{h} \in C_b^r(\mathbb{R}^n)$

and $D^\alpha \tilde{h}(x) = i^{|\alpha|} (\widetilde{\xi^\alpha h})(x)$.

Now use the hypothesis $f \in L^2(\mathbb{R}^n)$ to conclude $\tilde{h} = \tilde{\hat{f}} = f$. $\boxed{\xi}$

Proof of Sobolev :

By lemma VII. 32 we have $\xi^\alpha \hat{f} \in L^2(\mathbb{R}^n)$ $\forall\ |\alpha| \leq k$. In order to show that $f \in C_b^r(\mathbb{R}^n)$ it suffices to show that $\xi^\alpha \hat{f} \in L^1(\mathbb{R}^n)$ $\forall\ |\alpha| \leq r$. To this end, write $\xi^\alpha \hat{f} = h_1 \cdot h_{2,\alpha}$ where

$$h_1 = (1 + \|\xi\|^k)\hat{f}$$

$$h_{2,\alpha} = \frac{\xi^\alpha}{(1 + \|\xi\|^k)}.$$

And let's show that $h_1, h_2 \in L^2(\mathbb{R}^n)$.

This will then imply, $|\alpha| \leq r$.

$$\| \xi^\alpha \hat{f} \|_1 \leq \| h_1 \|_2 \, \| h_{2,\alpha} \|_2$$

Lemma $\overline{\text{VII}}$. 34 will then imply that

$f \in C_b^r(\mathbb{R}^n)$ and:

$$D^\alpha f = i^{|\alpha|} \widehat{(\xi^\alpha \hat{f})}$$

and thus $\| D^\alpha f \|_\infty \leq \| \xi^\alpha \hat{f} \|_1$.

On the other hand we will relate $\| h_1 \|_2$ to the Sobolev norm of f which will conclude the proof.

To estimate $\| h_1 \|_2$ use that

$$\| \xi \| \leq \sum_{j=1}^n |\xi_j| \text{ , hence}$$

$$\cancel{(1 + \| \xi \|^r)^2 \leq (1 + \| \xi \|)^{2r} \leq \left(1 + \sum_{j=1}^n |\xi_j|\right)^{2r}}$$

~~and Hölder's inequality applied to the RHS~~

$$\|\xi\| \leq \sum_{i=1}^{n} |\xi_i| \leq n^{1-1/k} \left(\sum_{i=1}^{n} |\xi_i|^k \right)^{1/k}$$

that is $\quad \|\xi\|^k \leq n^{k-1} \left(\sum_{i=1}^{n} |\xi_i|^k \right).$

Thus

$$|h_1| \leq n^{k-1} \left\{ |\hat{f}| + \sum_{i=1}^{n} |\xi_i^k \hat{f}| \right\}.$$

Which implies using Lemma $\overline{\text{VII}}.32$

and Plancherel:

$$\|h_1\|_2 \leq n^{k-1} \left\{ \|f\|_2 + \sum_{i=1}^{n} \|\partial_i^k f\|_2 \right\}$$

$$\leq n^{k-1} \|f\|_{2,k} .$$

Next: $\quad |h_{2,\alpha}(\xi)| \leq \dfrac{\|\xi\|^{|\alpha|}}{1+\|\xi\|^k} \leq \dfrac{\|\xi\|^r}{1+\|\xi\|^k}.$

But in polar coordinates

$$\int_{\mathbb{R}^n} \frac{\|\xi\|^{2r}}{(1+\|\xi\|^k)^2} = C_n \cdot \int_0^\infty dr\, r^{n-1} \frac{r^{2r}}{(1+r^k)^2}$$

which converges $\Longleftrightarrow$ $k > r + \frac{n}{2}$.

$\boxed{\xi}$