Here we present two applications of the Baire category theorem, the first is due to Baire:

<u>Thm IV.11</u> Let $f_n : X \to \mathbb{C}$ be a sequence of continuous functions on a complete metric space such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists $\forall x \in X$. Then the set of points where f is continuous is generic in X.

For a proof, see Stein–Shakarchi Chapter 4, 1.1.

It is well known that in $\mathbb{R}$ there are continuous functions that are nowhere differentiable. Example:

$$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i 2^n x}, \quad 0 < \alpha \leq 1$$

and the question is: how common is this phenomenon? In fact, let $C([0,1])$ be the Banach space of continuous functions with sup norm $\|f\|_{\infty} = \sup_{x \in [0,1]} |f(x)|$.

Then:

<u>Thm IV.12</u> The set of functions in $C([0,1])$ that are nowhere differentiable is generic.

Again, a proof can be found in Stein-Shakarchi: Chapter 4, 1.2.

In fact while both theorems use the Baire category theorem, the proofs are rather tricky.

We close this subsection with an application of Baire category which will have far reaching consequences in Functional Analysis.

<u>Proposition IV.13</u> : (Principle of uniform boundedness)

Let (X, d) be a complete metric space and $f_\lambda : X \longrightarrow \mathbb{R}$, $\lambda \in \Lambda$ a family of continuous functions such that:

$$\sup_{\lambda \in \Lambda} |f_\lambda(x)| < +\infty \quad \forall x \in X.$$

Then there is a ball $B := B_{<r}(y)$, $r > 0$, such that:

$$\sup_{\lambda \in \Lambda} \sup_{x \in B} |f_\lambda(x)| < +\infty .$$

<u>Proof</u>: For every $n \in \mathbb{N}$ consider the closed subset

$$A_n := \{ x \in X : |f_\lambda(x)| \leq n \quad \forall \lambda \in \Lambda \}$$

$$= \bigcap_\lambda \{ x \in X : |f_\lambda(x)| \leq n \}$$

Then by hypothesis $X = \bigcup_{n \in \mathbb{N}} A_n$ and by Thm IV.5 (3) there exists $n_0 \in \mathbb{N}$ with $\mathring{A}_{n_0} \neq \emptyset$. Now take $y \in \mathring{A}_{n_0}$ and $r > 0$ with $B_{<r}(y) \subset A_{n_0}$.

$\blacksquare$

IV. 2. The uniform boundedness principle.

The combination of Prop. IV. 13 with the linear structure has the following consequence:

Thm. IV. 14. (Banach–Steinhaus) Let $(V, \|\ \|_V)$ be a Banach space, $(W, \|\ \|_W)$ a normed space and $T_\lambda \in \mathcal{B}(V, W)$, $\lambda \in \Lambda$, a family of bounded linear operators with

$$\sup_{\lambda \in \Lambda} \|T_\lambda(u)\|_W < +\infty \quad \forall u \in V.$$

Then:

$$\sup_{\lambda \in \Lambda} \|T_\lambda\| < +\infty$$

<u>Proof</u>: Let $f_\lambda : V \to \mathbb{R}$ be defined

by $$f_\lambda(v) := \|T_\lambda(v)\|_W \quad , \quad \lambda \in \Lambda,$$

Then f_λ is continuous and $\forall v \in V$

$$\sup_\lambda f_\lambda(v) < +\infty.$$

By Prop. IV. 13, since V is a Banach space, there exists $w \in V$ ~~—~~, $r > 0$, and a constant $C \geq 0$ such that

$$\|T_\lambda(v)\|_W \leq C \qquad \forall v \in B_{<r}(w).$$

That is: $\|T_\lambda(w+u)\|_W \leq C \qquad \forall u \in B_{<r}(0)$

Since $\|T_\lambda(w) + T_\lambda(u)\| \geq \|T_\lambda(u)\| - \|T_\lambda(w)\|$

We get: $$\|T_\lambda(u)\| \leq C + \|T_\lambda(w)\|$$
$$\leq 2C$$

which implies $\|T_\lambda(x)\| \leq \dfrac{2C}{r} \qquad \forall x \in B_{<1}(0)$
$\forall \lambda \in \Lambda$

hence
$$\sup_{\lambda \in \Lambda} \| T_\lambda \| < +\infty .$$
$\boxed{\text{L}}$

Our first application is to the question, What can on say about a sequence
$$T_n : V \to W$$
of bounded operators converging pointwise?

<u>Corollary IV.15</u> Let $T_n \in \mathcal{B}(V, W)$ where V is a Banach space and assume
$$T(x) = \lim_{n \to \infty} T_n(x)$$
exists $\forall x \in V$. Then:

(a) $\sup_{n \geq 1} \| T_n \| < +\infty$.

(b) $T \in \mathcal{B}(V, W)$

(c) $\| T \| \leq \liminf_{n \to \infty} \| T_n \|$.

Remark IV.16 The theorem does NOT say that $\|T - T_n\| \to 0$. Indeed consider $T_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$x = \sum_{k=0}^{\infty} x_k \delta_k \quad , x \in \ell^2(\mathbb{N})$$

$$T_n(x) = x_n \delta_n .$$

Then $\|T_n(x)\| = |x_n| \longrightarrow 0 \quad \forall n \in \mathbb{N}$.

So $\lim_{n \to \infty} T_n(x) = 0$.

But $\|T_n\| = 1 \quad \forall n \in \mathbb{N}$.

Proof of ~~Thm IV~~ Corollary IV.15 :

(a) Since $(T_n(x))_{n \geq 1}$ converges $\forall x$,

We have $\sup_{n \geq 1} \|T_n(x)\| < +\infty \quad \forall x \in V$

which by Thm IV.14 implies

$$\sup_{n \geq 1} \|T_n\| < +\infty .$$

(b)+(c): Clearly T is linear. Let

$K := \liminf_{n\to\infty} \|T_n\|$ and $(n_k)_{k\geq 1}$

with $n_k \to \infty$ and $K = \lim_{k\to\infty} \|T_{n_k}\|$.

$\forall x \in V$: $T(x) = \lim_{k\to\infty} T_{n_k}(x)$, hence

$$\|T(x)\| = \lim_{k\to\infty} \|T_{n_k}(x)\| \leq \lim_{k\to\infty} \|T_{n_k}\| \|x\|$$

which implies $\|T\| \leq K$. $\square$

Next we deduce two corollaries that are useful to detect bounded subsets in Banach spaces.

~~Corollary IV.17 Let E be a Banach space and B ⊂ E a subset such that ∀λ ∈ E* λ(B) ⊂ K is bounded.~~

~~Then B ⊂ E is bounded.~~

<u>Cor. IV. 17</u> Let E be a normed space and $B \subset E$ a subset such that $\forall f \in E^*$, $f(B) \subset \mathbb{K}$ is bounded. Then $B \subset E$ is bounded.

<u>Proof</u>: We apply Thm IV. 14 with $V = E^*$, $W = \mathbb{K}$ and $\Lambda = B$. Define $\forall v \in B$,

$$T_v : E^* \longrightarrow \mathbb{K}$$
$$f \longmapsto f(v).$$

Then $\sup_{v \in B} |T_v(f)| < +\infty \quad \forall f \in E^*$.

By Thm IV. 14 we conclude that

$$\sup_{v \in B} \| T_v \| < +\infty .$$

But by Cor. II. 10 we have $\|T_v\| = \|v\|$ which concludes the proof. $\boxtimes$

We also have the dual statement:

Cor. IV.18 Assume E is a Banach space and $B^* \subset E^*$ is a subset such that $\{ f(x) : f \in B^* \} \subset \mathbb{K}$ is bounded $\forall\, x \in E$. Then

$B^* \subset E^*$ is bounded.

Proof: left as an exercise. ☒