VI.2. The Markov–Kakutani fixed point theorem.

Let E be a topological vector space:

Def. VI.17 An automorphism of E is a bijective continuous linear map $T: E \to E$ whose inverse $T^{-1}: E \to E$ is continuous.

Then the set $Aut(E)$ of automorphisms of E forms a group under composition.

Here are two interesting instances:

Example VI.18 Let X be compact Hausdorff, $E = M(X)$ the space of signed regular Borel measures with weak*-topology. Then we constructed in Chapter V a group homomorphism

$$\lambda^* : \mathcal{H}omeo(X) \longrightarrow \operatorname{Aut} E$$

which takes the concrete form:

$$\lambda^*(\Psi)(\mu) = \Psi_*(\mu).$$

We observed that the weak*–compact convex subset $M^1(X)$ of probability measures is invariant under $\lambda^*(\Psi)$ $\forall \Psi \in \mathcal{H}omeo(X)$.

__Example VI.19__ Let Γ be a group with discrete topology. A __mean__ on Γ is a continuous linear form $\mu \cancel{\lambda} \in \ell^\infty(\Gamma, \mathbb{R})^*$ such that

$$(1)\ \mu\cancel{\lambda}(\mathbb{1}_\Gamma) = 1$$

$$(2)\ \mu\cancel{\lambda}(f) \geq 0 \quad \forall f \geq 0,\ f \in \ell^\infty(\Gamma).$$

Then the set

$$\mathcal{M}(\Gamma) = \{ \underset{\mu}{\cancel{\lambda}} \in \ell^\infty(\Gamma, \mathbb{R})^* : \underset{\mu}{\cancel{\lambda}} \text{ is a mean} \}$$

is a convex, weak*-closed subset of the unit ball of $\ell^\infty(\Gamma, \mathbb{R})^*$ and hence compact.

Now $\forall \gamma \in \Gamma$, $g \in \ell^\infty(\Gamma, \mathbb{R})$, let:

$\lambda(\gamma) g(x) := g(\gamma^{-1} x)$. Then

$$(1) \ \|\lambda(\gamma) g\|_\infty = \|g\|_\infty \quad \forall \gamma \in \Gamma, \ \forall g \in \ell^\infty(\Gamma, \mathbb{R}).$$

$$(2) \ \lambda(\gamma_1 \gamma_2) = \lambda(\gamma_1)\lambda(\gamma_2)$$

Hence $\lambda(\gamma) : \ell^\infty(\Gamma, \mathbb{R})$ is a bijective isometry. Its adjoint

$$\lambda(\gamma)^* : \ell^\infty(\Gamma)^* \longrightarrow \ell^\infty(\Gamma)^*$$

is therefore weak*-continuous. Setting

$$\lambda^*(\gamma) := \left[\lambda(\gamma)^*\right]^{-1}$$ we obtain

a group homomorphism

$$\lambda^* : \Gamma \longrightarrow \mathrm{Aut}\left(\ell^\infty(\Gamma)^*\right).$$

If $m \in \ell^\infty(\Gamma)^*$ then $(\lambda^*(\gamma)m)(g) = m(\lambda(\gamma^{-1})g)$.

Clearly, if m is a mean, $\lambda^*(\gamma)m$ is a mean $\forall \gamma \in \Gamma$. Thus the weak*-compact, convex subset $\mathcal{M}(\Gamma)$ is invariant under $\lambda^*(\gamma)$, $\gamma \in \Gamma$.

Observe that if $m \in \mathcal{M}(\Gamma)$ we can define a set function $\mu : \mathcal{P}(\Gamma) \to \mathbb{R}_{\geq 0}$ by $\mu(E) := m(\chi_E)$. This set function has then the following properties

(1) $\mu(\Gamma) = 1$

(2) μ is finitely additive.

Theorem VI.20. (Markov-Kakutani) Let E be a TVS generated by a sufficient family of seminorms, G an abelian group and $\pi : G \to \mathrm{Aut}(E)$ a homomorphism. Assume that $A \subset E$ is compact, convex, $\neq \phi$, and G-invariant, that is $\pi(g)(A) \subset A$ $\forall g \in G$. Then there exists in A a point fixed by $\pi(g)$ $\forall g \in G$.

Proof: For every $g \in G$ and $n \geq 1$, define $M_{n,g} : E \to E$ by

$$x \longmapsto \frac{1}{n} \sum_{k=0}^{n-1} \pi(g^k)(x).$$

Then $M_{n,g}$ is a continuous linear map and since A is convex and $\pi(g^k)(A) \subset A$

We have $M_{n,g}(A) \subset A$.

Let

$$G^* := \{ M_{n_1, g_1} \circ \cdots \circ M_{n_\ell, g_\ell} : \ell \geq 1,$$
$$(n_1, \ldots, n_\ell) \in (\mathbb{N}_{\geq 1})^\ell, \ (g_1, \ldots, g_\ell) \in G^\ell \}.$$

This is a family of continuous linear maps $E \to E$ with the following properties:

(1) If $T, S \in G^*$ then $T \circ S \in G^*$.

(2) $T(A) \subset A \quad \forall T \in G^*$

(3) If $T, S \in G^*$ then $T \circ S = S \circ T$.

<u>Claim</u>: $\bigcap_{T \in G^*} T(A) \neq \emptyset$.

Now $T(A) \subset A$ is compact $\forall T \in G^*$ so that it suffices to show that $\forall T_1, \ldots, T_\ell \in G^*$: $\bigcap_{i=1}^{\ell} T_i(A) \neq \emptyset$.

But we have that $\forall\ 1 \leq i \leq l$:

$$T_i(A) \supset T_i(T_1 \cdots T_{i-1} T_{i+1} \cdots T_l)(A)$$

but since all the T_j's commute we have

$$T_i(T_1 \cdots T_{i-1} \circ T_{i+1} \cdots T_l) = T_1 \cdots T_l$$

and hence

$$\bigcap_{i=1}^{l} T_i(A) \supset T_1 \cdots T_l(A) \neq \phi.$$

This proves the claim.

Let now $y \in \bigcap_{T \in G^*} T(A)$. Then it

follows that $\forall n \geq 1$, $\forall g \in G$, $\exists x_{n,g} \in A$

with $M_{n,g}(x_{n,g}) = y$ that is

$$y = \frac{1}{n} \sum_{i=0}^{n-1} \pi(g^i) x_{n,g}.$$

Then $\pi(g)y - y = \frac{1}{n}\left\{\pi(g^n)x_{n,g} - x_{n,g}\right\}$.

Let $\{\|\ \|_\alpha : \alpha \in A\}$ be the sufficient family of seminorms defining the topology on E. Then $\forall \alpha \in A$:

$$\|\pi(g)y - y\|_\alpha \le \frac{1}{n}\left\{\|\pi(g^n)x_{n,g}\|_\alpha + \|x_{n,g}\|_\alpha\right\}.$$

Let $B_\alpha := \sup\{\|v\|_\alpha : v \in A\} < +\infty$, since A is compact. We conclude

$$\|\pi(g)y - y\|_\alpha \le \frac{2B_\alpha}{n} \qquad \forall n \ge 1$$

hence $\|\pi(g)y - y\|_\alpha = 0 \quad \forall \alpha \in A$

and hence $\pi(g)y = y \quad \forall g \in G.$ $\boxed{\xi}$

In the context of Example VI.18 we obtain the following corollaries, which we state in terms of group actions. Recall that a group action of a group G on a set X is a map

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g_* x$$

satisfying the following axioms:

(1) $e_* x = x \quad \forall x \in X$.

(2) $(g_1 g_2)_* x = (g_1)_* (g_{2*}(x)) \quad \forall g_1, g_2 \in G$
$\forall x \in X$.

If X is a topological space, the group action is by homeomorphisms if $\forall g \in G$ the map
$$\varphi_g : X \longrightarrow X$$
$$x \longmapsto g_* x$$
is a homeomorphism.

<u>Corollary VI.21</u> Let $G \times X \to X$ be an action by homeomorphisms of an <u>abelian</u> group G on a compact Hausdorff space X. Then there exists an invariant probability measure, that is, there is $\mu \in M^+(X)$

such that $(\Psi_g)_* \mu = \mu \quad \forall g \in G$.

<u>Proof</u>: Apply Thm VI.20 to the space $E = M(X)$ of signed regular Borel measures with weak*-topology and the homomorphism $\pi : G \longrightarrow \mathrm{Aut}(E)$ obtained by composing

$$\begin{array}{ccccc} G & \longrightarrow & \mathrm{Homeo}(X) & \xrightarrow{\lambda^*} & \mathrm{Aut}(E) \\ g & \longmapsto & \Psi_g & \longmapsto & \lambda^*(\Psi_g); \end{array}$$

Then the weak*-compact convex subset $M^1(X)$ is invariant under $\pi(g)$ $\forall g \in G$ and the corollary follows from Thm VI.20.

$\boxtimes$

<u>Corollary VI.22</u>: Let X be compact Hausdorff and $\Psi \in \mathrm{Homeo}(X)$. Then $\exists \mu \in M^1(X)$ with $\Psi_*(\mu) = \mu$.

<u>Proof</u>: Apply the preceding corollary to the group action:

$$\mathbb{Z} \times X \to X$$
$$(n, x) \mapsto \varphi^n(x). \qquad \boxed{\S}$$

In the context of Example VI.19, we obtain:

<u>Corollary VI.23</u> Let Γ be an abelian group. Then there exists a mean $m \in \mathcal{M}(\Gamma)$ that is invariant under $\lambda^*(\gamma)$, $\forall \gamma \in \Gamma$. In particular there exists a set function $\mu : \mathcal{P}(\Gamma) \to [0,1]$ with the properties

(1) $\mu(\Gamma) = 1$

(2) μ is finitely additive

(3) $\mu(\gamma E) = \mu(E)$, $\forall \gamma \in \Gamma$, $\forall E \subset \Gamma$.

This Corollary is the starting point of the theory of amenable groups: a group Γ is amenable if there is a mean $m \in \mathcal{M}(\Gamma)$ that is invariant under "left translations", that is $\lambda^*(\gamma) m = m$ $\forall \gamma \in \Gamma$. Not all groups are amenable; for instance the free group $\Gamma = \mathbb{F}_2(a, b)$ on two generators is not, and this is intimately connected to the paradoxical decomposition mentionned back in Thm II.21. (Banach–Tarski paradox).