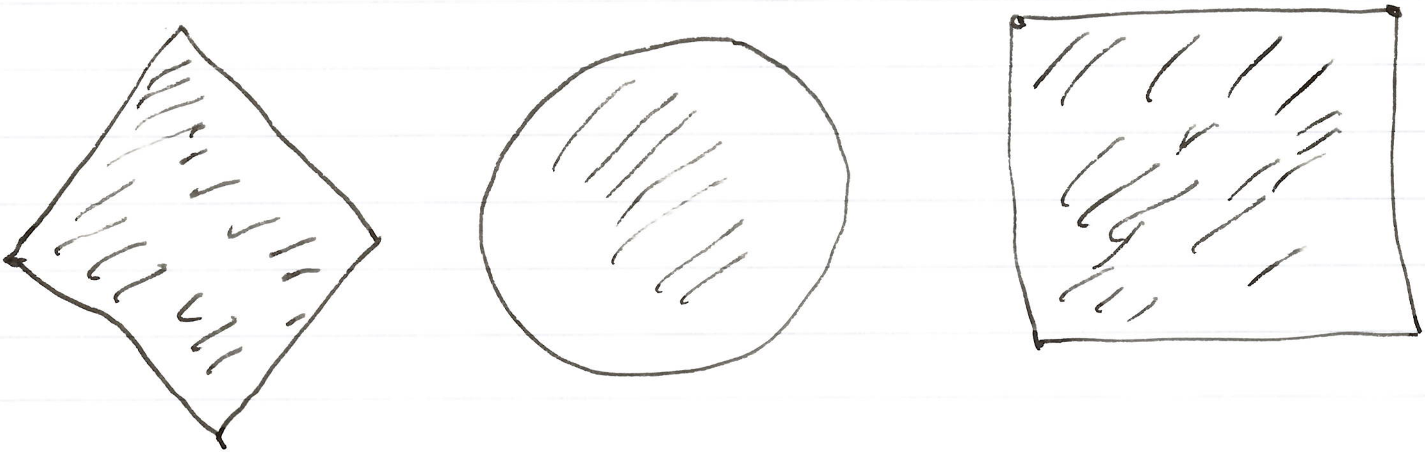
We deduce then the following

Corollary I.16. In a normed space, any finite dimensional subspace is closed.

Proof: Since on a f.d. space all norms are equivalent such a normed space is complete and hence closed as subspace of any normed space. $\boxtimes$

Even for equivalent norms their unit balls can have very different geometric properties. For instance on $\mathbb{R}^2$ and for $1 \leq p < \infty$, $\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_p = \sqrt[p]{|x_1|^p + |x_2|^p}$, while $\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_\infty = \max(|x_1|, |x_2|)$.

$p = 1$ $\qquad$ $p = 2$ $\qquad$ $p = \infty$



I.2. Continuous Linear Maps

Having defined the objects of the category of normed spaces, we need the morphisms. These turn out to be continuous linear maps and admit various characterizations.

In a normed space $(V, \| \|)$ we say that a subset $B \subset V$ is bounded if there is $0 \leq R < +\infty$ such that $B \subset B_{\leq R}(0)$.

Def. I.17: A linear map $T : V \to W$ between normed spaces $(V, \| \|_V)$, $(W, \| \|_W)$ is bounded if $T(B)$ is bounded whenever $B \subset V$ is bounded.

Observe that the property for T to be bounded is equivalent to:

$$\|T\| := \sup \{ \|T(x)\|_W : \|x\|_V \leq 1 \} < +\infty. \qquad (I.17^*)$$

We have then

<u>Thm I.18</u> : Let $T : V \to W$ be a linear map of normed spaces $(V, \| \|_V)$, $(W, \| \|_W)$. The following are equivalent:

(1) T is continuous in $0 \in V$

(2) T is continuous on V

(3) T is bounded.

(4) T is Lipschitz continuous with Lipschitz constant $\|T\|$.

<u>Proof</u> :

(1) $\Rightarrow$ (2) It follows from lemma I.3 (2) that $\forall u \in V$, $L_u : V \to V$, $x \mapsto x + u$

is continuous. The additivity of T can be expressed by the commutativity of the diagramm,

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "L_{-0}"'] & W \\ V \arrow[r, "T"'] & W \arrow[u, "L_{T(0)}"'] \end{tikzcd}$$

Thus $T \cancel{= L_{-0} \circ T \circ L_{T(0)}}$

$$= L_{T(0)} \circ T \circ L_{-0} \qquad (*)$$

If now T is continuous in 0, and $L_{-0}(0)=0$ this implies that $L_{T(0)} \circ T \circ L_{-0}$ is continuous at 0 and by $(*)$ that T is continuous at 0.

(1) $\Rightarrow$ (3) $B^{W}_{\leq 1}(0)$, the closed unit ball centered at $0 \in W$ is a neighborhood of 0. Since $T(0)=0$ and T is continuous at 0

there is $\varepsilon > 0$ such that $T(B^V_{\leq \varepsilon}(0)) \subset B^W_{\leq 1}(0)$

which implies $T(B^V_{\leq 1}(0)) \subset B^W_{\leq 1/\varepsilon}(0)$

and shows (3).

$(3) \Rightarrow (4)$: $\|T(x) - T(y)\|_W = \|T(x-y)\|_W$

$\leq \|T\| \cdot \|x-y\|_V$.

$(4) \Rightarrow (1)$ is obvious. $\boxed{\xi}$

From now on, given normed space $(V, \|\ \|_V)$

and $(W, \|\ \|_W)$ we will denote

$$B(V,W) = \{T : V \to W : \text{is linear and bounded}\}$$

With the Definition in I. 17* we have

<u>Prop. I. 19</u>: Let $U \xrightarrow{T} V \xrightarrow{S} W$ be

bounded linear maps between normed spaces,

then $\|S \cdot T\| \leq \|S\| \cdot \|T\|$.

<u>Proof</u>: For $x \in V$ with $\|x\|_V \leq 1$:

$$\| S \circ T(x) \| = \| S(T(x)) \| \leq \|S\| \cdot \|T(x)\|$$
$$\leq (\|S\| \, \|T\| \, \|x\|)$$
$$\leq \|S\| \, \|T\|$$

from which $\|S \circ T\| \leq \|S\| \cdot \|T\|$ follows.

$\boxed{B}$

<u>Remark I. 20</u> This simple observation is quite fundamental as it endows $\mathcal{B}(V, V)$ with the structure of Banach Algebra, which will be the topic of FA II.

Observe that if $(V, \| \|_V)$ and $(W, \| \|_W)$ are normed spaces then $T \mapsto \|T\|$ gives a norm on $\mathcal{B}(V, W)$.

<u>Def. I. 21</u> $\mathcal{B}(V, \mathbb{K})$ is called the dual space of V and denoted V^*.

<u>Prop. I.22</u>: If $(W, \| \|_W)$ is a Banach space then $(\mathcal{B}(V, W), \| \|)$ is a Banach space.

In particular for any normed space $(V, \| \|_V)$, V^* is a Banach space.

<u>Proof</u>: Let $(A_n)_{n \geq 1}$ be a Cauchy sequence in $\mathcal{B}(V, W)$. Since $\forall x \in V$,

$$\| A_n(x) - A_m(x) \| \leq \| A_n - A_m \| \cdot \| x \|,$$

$(A_n(x))_{n \geq 1}$ is a Cauchy sequence in W.

Hence ~~converges~~ converges and we define

$$A(x) := \lim_{n \to \infty} A_n(x) \quad \forall x \in V.$$

The fact that A is linear is an easy verification.

Next, it follows from

$$\big| \| A_n \| - \| A_m \| \big| \leq \| A_n - A_m \|$$

that $\lim_{n \to \infty} \| A_n \|$ exists and thus:

$$\| A(x) \| \leq \left(\lim_{n \to \infty} \|A_n\| \right) \|x\|$$

which implies that A is bounded.

Finally,

$$\| A(x) - A_n(x) \| = \lim_{l \to \infty} \| A_l(x) - A_n(x) \|$$

$$\leq \limsup_{l \to \infty} \| A_l - A_n \| \, \|x\|$$

which implies

$$\| A - A_n \| \leq \limsup_{l \to \infty} \| A_l - A_n \|$$

and since $(A_n)_{n \geq 1}$ is Cauchy

$$\limsup_{n \to \infty} \| A - A_n \| \leq \limsup_{n \to \infty} \limsup_{l \to \infty} \| A_l - A_n \| = 0$$

which implies $\lim_{n \to \infty} \| A - A_n \| = 0$ since

$\| A - A_n \| \geq 0 \quad \forall n \geq 1.$ $\boxed{\text{W}}$

In certain situations linear maps are automatically continuous;

<u>Proposition I.23</u> Let $(V, \|\ \|_V)$, $(W, \|\ \|_W)$ be normed spaces and $T: V \to W$ a linear map. Assume V is finite dimensional, then T is bounded.

<u>Proof</u>: Define for all $v \in V$:

$$\|v\| := \|v\|_V + \|T(v)\|_W .$$

One verifies easily that $\|\ \|$ is a norm on V; since V is finite dimensional, all norms on V are equivalent and (Prop. I.15) in particular there is $C > 0$ with $\|v\| \leq C \cdot \|v\|_V$.

This implies $\|T(v)\|_W \leq (C-1)\|v\|_V \quad \forall v \in V.$

In particular $C-1 \geq 0$; if $C-1=0$ then $T=0$ in particular bounded, and if $C-1>0$ T is bounded and we are done. $\boxed{\equiv}$

Before we move to examples of bounded linear maps we define the operation of adjunction which in a sense generalizes the transposition of a matrix.

Recall that if $(V, \|\,\|_V)$ is a normed $\mathbb{K}$-vector space, its dual V^* is defined as: $V^* = \mathcal{B}(V, \mathbb{K})$. Given now $A \in \mathcal{B}(V, W)$ a bounded linear map of normed $\mathbb{K}$-vector spaces, we have that $\forall\ \lambda \in W^*$, the composition $\lambda \circ A$,

$$V \xrightarrow{A} W \xrightarrow{\lambda} \mathbb{K}$$

defines an element in V^* denoted $A^*(\lambda)$.

This way we obtain a linear map

$$A^* : W^* \longrightarrow V^*$$

called the adjoint of A. Let's show that A^* is bounded:

$$|A^*(\lambda)(x)| = |\lambda(A(x))| \leq \|\lambda\| \, \|A\| \, \|x\|$$

which implies

$$\|A^*(\lambda)\| \leq \|A\| \cdot \|\lambda\|$$

and taking sup over $\|\lambda\| \leq 1$ gives

$$\|A^*\| \leq \|A\|.$$

We will see later as a consequence of Hahn-Banach, that in fact $\|A^*\| = \|A\|$. For the moment being we don't even know whether $V^* \neq 0$.

Assume $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces; we know from Analysis IV that the map

$$\begin{aligned} i_1 : \mathcal{H}_1 &\longrightarrow \mathcal{H}_1^* \\ v &\longmapsto i(v) \end{aligned}$$

where $i(v)(w) = \langle w, v \rangle$ is a bijective map. Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $T^* : \mathcal{H}_2^* \longrightarrow \mathcal{H}_1^*$. Define

$$\begin{tikzcd} \mathcal{H}_2 \arrow[r, "i_2"] \arrow[d, "T'"'] & \mathcal{H}_2^* \arrow[d, "T^*"] \\ \mathcal{H}_1 \arrow[r, "i_1"'] & \mathcal{H}_1^* \end{tikzcd}$$

that is: $T' := i_1^{-1} T^* i_2$.

Then $T' \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and verifies

$$\langle Tv, w \rangle = \langle v, T'w \rangle$$

$\forall v \in \mathcal{H}_1$,
$\forall w \in \mathcal{H}_2$.