IV. 3. The open mapping theorem and the closed graph theorem.

A general question, once one has a category ~~with~~ is whether bijective morphisms are automatically isomorphisms. In our case this translates to the question whether a bounded linear operator between normed spaces that is bijective has a bounded inverse. In general, no. But if both spaces are Banach, the answer is yes as will follow from the more general:

<u>Thm IV. 19</u> (open mapping thm.)

Let V, W be Banach spaces and

$$T : V \to W$$

a bounded <u>surjective</u> operator. Then $\exists\, c > 0$ with

$$T\left(B^V_{<1}(0)\right) \supset B^W_{<c}(0).$$

In particular T is an open mapping.

<u>Proof</u>:

(1) Claim: $\exists\, \varepsilon > 0$ such that

$$\overline{T\left(B_{<1}(0)\right)} \supset B_{\varepsilon}(0).$$

Since T is surjective, $\forall\, w \in W$, $\exists\, n \geq 1$ with $w \in T(B_{<n}(0)) \subset \overline{T(B_{<n}(0))}$ and thus $W = \bigcup_{n \geq 1} \overline{T(B_{<n}(0))}$.

By Baire Category Theorem $\exists\, n \geq 1$ with $\overset{\circ}{\overline{T(B_{<n}(0))}} \neq \emptyset$.

Hence $\overset{\circ}{\overline{T(B_{<1}(0))}} \neq \emptyset$.

Let $\varepsilon > 0$ and $x \in W$ with

$$\overline{T(B_{<1}(0))} \supset B(x, 2\varepsilon).$$

Here and elsewhere it will be convenient to have the following definitions: for subsets $S_1, S_2 \subset W$,

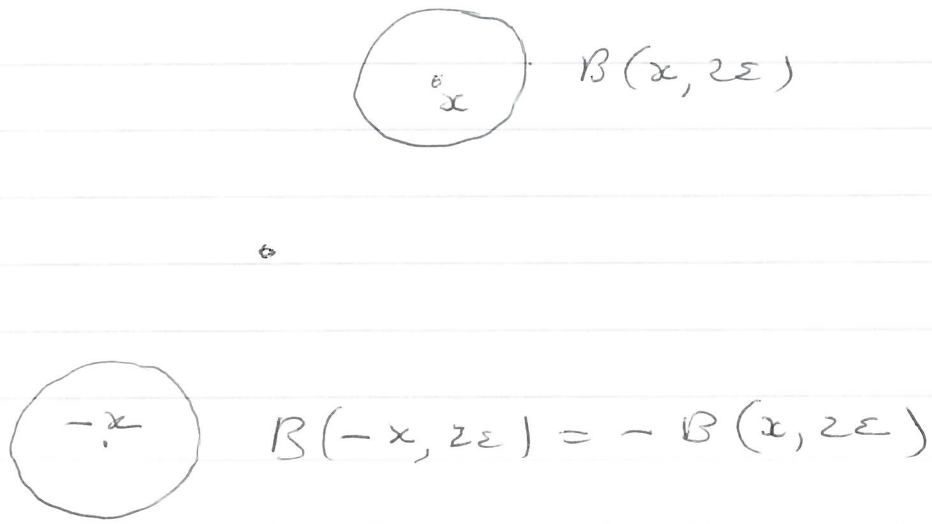
$$S_1 + S_2 = \{ s_1 + s_2 : s_i \in S_i \}$$

$$S_1 - S_2 = \{ s_1 - s_2 : s_i \in S_i \}$$

Then we have

$$\overline{T(B_{<1}(0))} + \overline{T(B_{<1}(0))} = \overline{T(B_{<1}(0))} - \overline{T(B_{<1}(0))}$$

$$\supset B(x, 2\varepsilon) - B(x, 2\varepsilon).$$



Now observe $B(x, 2\varepsilon) - B(x, 2\varepsilon)$

$$= B(x, 2\varepsilon) + B(-x, 2\varepsilon) \supset B(0, 2\varepsilon).$$

Next observe that if $(v_n)_{n \geq 1}$, $(w_n)_{n \geq 1}$ are sequences in $B_{<1}(0)$ such that $(T(w_n))$ and $T(w_n)$ converge then

$$T(v_n) + T(w_n) = T(v_n + w_n)$$

converges and $v_n + w_n \in B_{<2}(0)$. This implies

$$\overline{T(B_{<2}(0))} \supset \overline{T(B_{<1}(0))} + \overline{T(B_{<1}(0))}$$

and hence, putting everything together.

$$2\,\overline{T(B_{<1}(0))} \supset B(0,2\varepsilon)$$

hence $\overline{T(B_{<1}(0))} \supset B(0,\varepsilon)$,

(2) Let now $\varepsilon = 2c$, so that $\overline{T(B_{<1}(0))} \supset B(0,2c)$.

Then we claim: $T(B_{<1}(0)) \supset B(0,c)$.

We are going to use that:

$$\overline{T\left(B_{<1/2^n}(0)\right)} \supset B\left(0, \frac{c}{2^{n-1}}\right), \quad n \geq 1.$$

Let $y \in B(0,c) \subset \overline{T(B_{<1/2}(0))}$

and $z_1 \in B_{<1/2}(0)$ with $\| y - T(z_1) \| < \frac{c}{2}$.

Now $y - T(z_1) \in B_{<c/2}(0) \subset \overline{T(B_{<1/4}(0))}$

and we pick $z_2 \in B_{<1/4}(0)$ with

$$\| y - T(z_1) - T(z_2) \| < \frac{c}{4}.$$

In this way we construct a sequence $(z_n)_{n\geq 1}$ with: $z_n \in B_{<1/2^n}(0)$ and

$$\| y - (T(z_1) + \cdots + T(z_n)) \| < \frac{c}{2^n}.$$

Let $x_n := z_1 + \cdots + z_n$. Then clearly $(x_n)_{n\geq 1}$ is a Cauchy sequence in V and since V is Banach,

$$x := \lim_{n\to\infty} x_n \quad \text{exists.}$$

From the continuity of T we get

$$T(x) = y$$

and $\|x\| \leq |z_1| + \cdots + |z_n| < \frac{1}{2} + \cdots + \frac{1}{2^n} < 1.$

and $x \in B_{<1}(0)$.

(3) T is an open mapping: let $\Omega \subset V$ be any open set; $\forall v \in \Omega$, there is $\varepsilon > 0$

with $B_{<\varepsilon}(v) = v + B_{<\varepsilon}(0) \subset \Omega$.

Hence

$$B_{<\varepsilon c}(T(v)) = T(v) + B^W_{<\varepsilon c}(0) \subset T(v) + T(B^V_{<\varepsilon}(0))$$

$$= T(v + B^V_{<\varepsilon}(0))$$

$$= T(B_{<\varepsilon}(v)) \subset T(\Omega).$$

Hence $T(\Omega)$ is open.

$\boxed{\xi}$

<u>Corollary IV.20</u> Let $T : V \to W$ be a bounded linear operator between Banach spaces that is bijective. Then $T^{-1} : W \to V$ is bounded.

<u>Proof</u>: $T^{-1} : W \to V$ is well defined and by Thm IV.19 T is open, hence T^{-1} is continuous

$\boxed{\xi}$

And:

Corollary IV.21 Assume V is a vector space endowed with two norms $\| \ \|_1$, $\| \ \|_2$ such that $(V, \| \ \|_1)$, $(V, \| \ \|_2)$ are Banach. Assume $\exists\, c > 0$ such that $\|v\|_2 \leq c \|v\|_1$ $\forall v \in V$. Then there is $k > 0$ such that

$$\|v\|_1 \leq k \|v\|_2 \quad \forall v \in V.$$

Proof: Left as an exercise.

Next we turn to a rather astonishing consequence of Corollary IV.20:

Thm IV.22 (Closed graph theorem).
Let $T : V \to W$ be a linear map between Banach spaces V, W. Assume that

$$\text{graph}(T) = \{(v, T(v)) : v \in V\}$$

is closed in $V \times W$. Then T is bounded.

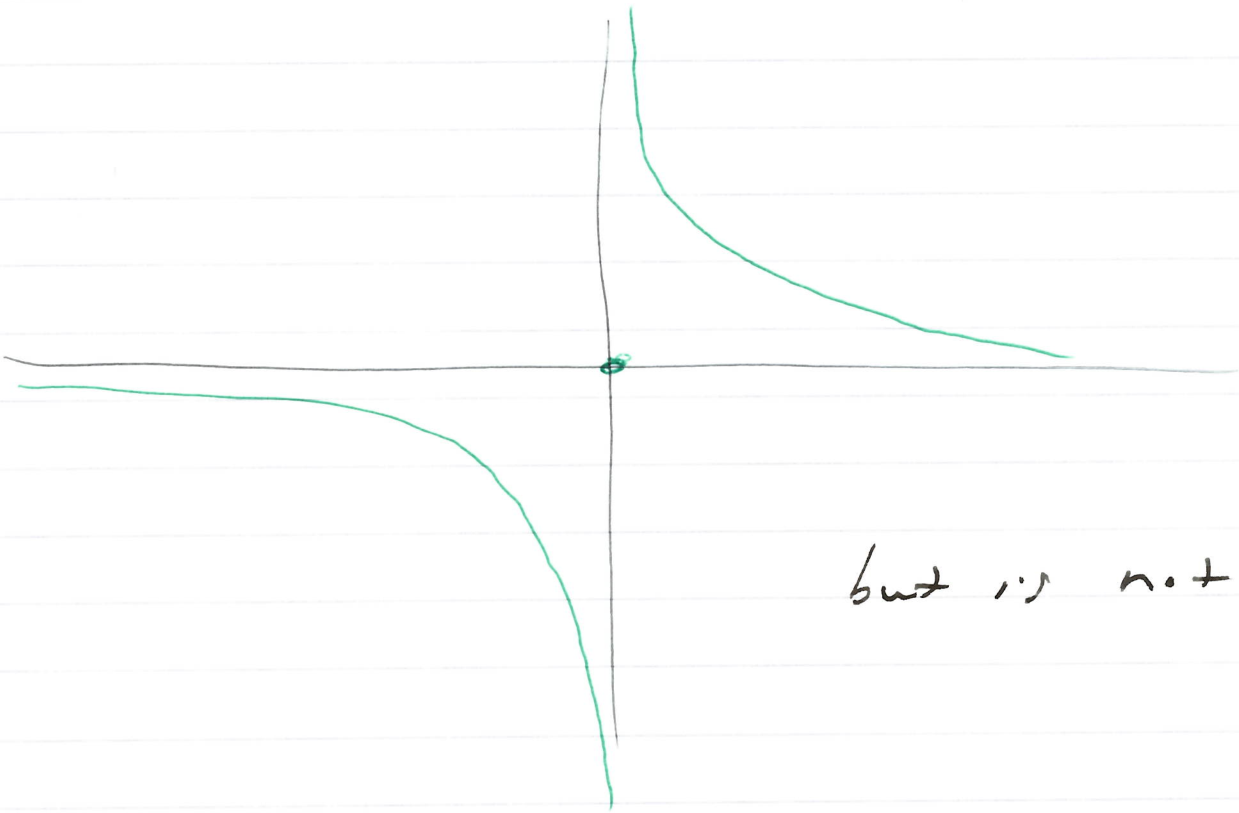
Remarks IV.23

(1) The converse holds since W is Hausdorff.

(2) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ $x \neq 0$

$f(0) = 0$

as a closed graph:



but is not continuous.

<u>Proof:</u> $V \times W$ with the norm $\|(v,w)\| = \|v\|_V + \|w\|_W$ is a Banach space and so is $\mathrm{graph}(T) \subset V \times W$ since it is closed. Observe that the projections $p_V : V \times W \to V$, $p_W : V \times W \to W$ are continuous, linear. Now:

$$p_V|_{\mathrm{graph}(T)} : \mathrm{graph}(T) \longrightarrow V$$

is a continuous linear bijection, hence by Cor. IV.20

$$\left(p_V|_{\mathrm{graph}(T)}\right)^{-1} : V \longrightarrow \mathrm{graph}(T)$$

is continuous as well. Since T is the composition of $\left(p_V|_{\mathrm{graph}(T)}\right)^{-1}$ with p_W it is continuous.

$\boxed{\xi}$

<u>Remark IV.24</u> Let $C([0,1])$ and $C^1([0,1])$ both be endowed with the sup norm $\| \ \|_b$.

The derivative
$$C^1([0,1]) \longrightarrow C([0,1])$$
$$f \longmapsto f'$$
is a linear map; but it is not bounded.

By a well known fact in Analysis I, its graph is closed in $C^1([0,1]) \times C([0,1])$.

Observe that since $C^1([0,1])$ is not complete for $\| \ \|_b$, the closed graph theorem does not apply, and indeed fails.

IV.4. Grothendieck's theorem on closed subspaces of L^p.

Here we present a quite non-trivial application of the closed graph theorem, namely:

Thm IV.25 Let $(X, \mathcal{F}, \mu)$ be a finite measure space, that is $\mu(X) < +\infty$.

Suppose that

(1) E is a closed subspace of $L^p(X, \mu)$ for some $1 \leq p < +\infty$.

(2) $E \subset L^\infty(X, \mu)$.

Then E is finite dimensional.

<u>Proof</u>: Equipped with the L^p-norm, E is a Banach space. Let

$$I : E \to L^\infty(X, \mu)$$

be the identity map, $I(f) = f$, $f \in E$. We claim that the graph of I is closed: indeed assume $f_n \to f$ in L^p and

$$f_n \to g \text{ in } L^\infty.$$

By Thm I.9 there is a subsequence $(f_{n_k})_{k \geq 1}$ that converges to f almost everywhere. As a result $f = g$. By the closed graph theorem there is $M > 0$ with

$$\|f\|_\infty \leq M \|f\|_p \quad \forall f \in E.$$

<u>Claim</u>: $\exists A > 0$ such that $\|f\|_\infty \leq A \|f\|_2$ $\forall f \in E$.

Observe that since $E \subset L^\infty(X, \mu)$, $\forall f \in E$

$$\|f\|_2^2 = \int_X |f(x)|^2 d\mu(x) \leq \|f\|_\infty^2 \cdot \mu(X)$$

and since $\mu(X) < +\infty$, $\|f\|_2 < +\infty$ $\forall f \in E$.

Now to prove the claim assume first $1 \leq p \leq 2$: apply Hölders inequality with the conjugate exponents $r = \frac{2}{p}$, $r^* = \frac{2}{2-p}$:

$$\int |f(x)|^p \cdot 1 \, d\mu(x) \leq \left(\int |f(x)|^2 d\mu(x) \right)^{p/2} \left(\int_X 1 \, d\mu(x) \right)^{\frac{2-p}{2}}$$

thus $\|f\|_p \leq \|f\|_2 \, \mu(X)^{\frac{2-p}{2p}}$.

Now assume $2 < p < +\infty$: notice that

$$|f(x)|^p \leq \|f\|_\infty^{p-2} |f(x)|^2$$

and integrating this inequality gives

$$\|f\|_p^p \leq \|f\|_\infty^{p-2} \|f\|_2^2$$

We now use $\|f\|_\infty \leq M \|f\|_p$, $f \in E$

and deduce $\|f\|_p^p \leq M^{p-2} \|f\|_p^{p-2} \|f\|_2^2$

from which

$$\|f\|_p \leq M^{\frac{p-2}{2}} \|f\|_2, \forall f \in E$$

follows.

Now we return to the proof of Thm IV. 25.

Let $f_1, \dots, f_n$ be an orthonormal set in E. If $\dim E \geq n$ such a set can be obtained by the Gram-Schmidt orthonormalization procedure.

Let $\mathbb{B} = \{ \varsigma = (\varsigma_1, .., \varsigma_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |\varsigma_j|^2 \leq 1 \}$

be the unit ball in $\mathbb{C}^n$ and for

every $\varsigma \in \mathbb{B}$, let $f_\varsigma(x) = \sum_{j=1}^{n} \varsigma_j f_j(x)$.

Then $\| f_\varsigma \|_2 \leq 1$ and by the claim

we deduce

$$\| f_\varsigma \|_\infty \leq A , \quad \forall \varsigma \in \mathbb{B}.$$

So for every $\varsigma \in \mathbb{B}$ there exists a meas. subset $X_\varsigma \subset X$ with $\mu(X_\varsigma) = \mu(X)$ such that $|f_\varsigma(x)| \leq A \quad \forall x \in X_\varsigma$.

Let now $\{ \varsigma_n : n \geq 1 \} \subset \mathbb{B}$ be a countable dense subset of $\mathbb{B}$ and $S := \bigcap_{i=1}^{\infty} X_{\varsigma_i}$.

Then $|f_{\varsigma_i}(x)| \leq A \quad \forall x \in S, \forall i \in \mathbb{N}$

and $\mu(S) = \mu(X)$. But observe that

$\forall x \in S$, $\varsigma \mapsto f_\varsigma(x)$ is continuous,

and hence $|f_\xi(x)| \leq A \quad \forall x \in S \quad \forall \xi \in \mathbb{B}$.

From this, we claim that

$$(*) \qquad \sum_{j=1}^{n} |f_j(x)|^2 \leq A^2 \quad \forall x \in S.$$

Indeed, we may assume that the left hand side is non-zero; then if we let $\sigma := \left(\sum_{j=1}^{n} |f_j(x)|^2 \right)^{1/2}$ and set $\xi_j := \overline{f_j(x)}/\sigma$, $|f_\xi(x)| \leq A$ implies

$$\frac{1}{\sigma} \sum_{j=1}^{n} |f_j(x)|^2 \leq A$$

as we claimed.

Finally integrating $(*)$ over X we find $\quad n \leq A^2 \cdot \mu(X)$. $\quad \blacksquare$