28.9.23

In the case of Hilbert spaces we will denote T' by T^* (by abuse of notation).

Let $\mathcal{H}$ be a Hilbert space, a bounded operator $T: \mathcal{H} \to \mathcal{H}$ is called

<u>Def. I.24</u> (a) Self-adjoint if $T^* = T$

(b) Unitary if $T^* T = T T^* = \mathrm{Id}_{\mathcal{H}}$.

<u>Remark I.25</u> A unitary operator has in particular the property that $\|Tv\|^2 = \|v\|^2$ $\forall v \in \mathcal{H}$.

More generally:

<u>Def. I.26</u> A bounded operator $T: V \to W$ of normed spaces is an <u>isometry</u> if

$$\|Tu\|_W = \|u\|_V \quad \forall u \in V.$$

<u>Example I.27</u> (Multiplication Operator).

Let $(X, \mathcal{F}, \mu)$ be a measure space (see Example I.8) and $\varphi \in L^\infty(X)$. Then if $f \in L^p(X, \mu, \mathbb{K}) = L^p(X)$ then

$$|f(x) \cdot \varphi(x)| \leq |f(x)| \cdot \|\varphi\|_\infty \qquad (*)$$

and hence $f \cdot \varphi \in L^p(X)$. We deduce from $(*)$ that the linear operator

$$M_\varphi : L^p(X) \longrightarrow L^p(X)$$
$$f \longmapsto f \cdot \varphi$$

is bounded with $\|M_\varphi\| \leq \|\varphi\|_\infty$.

In fact $\|M_\varphi\|_{\infty} = \|\varphi\|_\infty$ (exercise).

This example is absolutely fundamental!

here is why: let $(V, \langle\ ,\ \rangle)$ be a finite dimensional $\mathbb{R}$-inner product space.

or in other words a finite dimensional real Hilbert space. Let $T: V \to V$ be a self adjoint map, $n = \dim V$ and $X = \{1, 2, \dots, n\}$ with μ = counting measure.

Then there exists a Hilbert space isomorphism

$$V \xrightarrow{\delta} \ell^2(X)$$

~~such~~ and $\varphi \in L^\infty(X, \mathbb{R})$ such that

$$\begin{tikzcd}
V \arrow[r, "\delta"] \arrow[d, "T"'] & \ell^2(X) \arrow[d, "M_\varphi"] \\
V \arrow[r, "\delta"'] & \ell^2(X)
\end{tikzcd}$$

commutes.

This is a reformulation of the theorem that a real symmetric matrix has an orthonormal set of eigenvectors with real eigenvalues.

Example I. 28 (unitary representation)

Let Γ be a group which we consider as a measure space with counting measure.

In this example $l^2(\Gamma) = \ell^2(\Gamma, \mathbb{C})$.

For $f \in l^2(\Gamma)$ and $\gamma \in \Gamma$ define

$$(\lambda(\gamma) f)(\eta) := f(\gamma^{-1}\eta). \qquad (*)$$

Then

$$\langle \lambda(\gamma) f, g \rangle = \sum_{\eta} f(\gamma^{-1}\eta)\overline{g(\eta)}$$

$$= \sum_{\eta} f(\eta)\overline{g(\gamma\eta)} = \langle f, \lambda(\gamma^{-1}) g \rangle$$

From which we deduce:

$$\lambda(\gamma)^* = \lambda(\gamma^{-1}).$$

In addition the definition in $(*)$ with the inverse of γ is chosen such that

$$\lambda(\gamma_1 \gamma_2) = \lambda(\gamma_1)\lambda(\gamma_2).$$

In particular:

$$\lambda(\gamma)^* \lambda(\gamma) = \lambda(\gamma^{-1})\lambda(\gamma) = \lambda(e) = \mathrm{Id}$$

$$\lambda(\gamma)\lambda(\gamma)^* = \lambda(\gamma)\,\lambda(\gamma^{-1}) = \lambda(e) = \mathrm{Id}$$

thus $\lambda : \Gamma \longrightarrow U(\ell^2(\Gamma))$ is a homomorphism into the <u>group</u> $U(\ell^2(\Gamma))$ of unitary operators of $\ell^2(\Gamma)$.

<u>Exercise</u>: ~~...~~ for $\gamma \in \Gamma$, $\lambda(\gamma)$ has an eigenvector in $\ell^2(\Gamma) \Longleftrightarrow \gamma$ is of finite order in Γ.

<u>Example I.29</u> (Integral Operators)

Let $(X, \mu, \mathcal{F})$ be a σ-finite measure space. This means that X is a countable union of measurable sets of finite measure and hence Fubini's theorem holds.

Let $K \in L^2(X \times X, \mu \times \mu; \mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then

$$\iint_{X \times X} |K(x,y)|^2 \, d\mu(x) \, d\mu(y) < +\infty$$

and hence by Fubini's theorem we have that for almost every $x \in X$

$$\int_X |K(x,y)|^2 \, d\mu(y) < +\infty$$

that is, $y \longmapsto K(x,y)$ is in $L^2(X,\mu)$ and thus $\forall f \in L^2(X,\mu)$

$$T_K f(x) = \int_X f(y) K(x,y) \, d\mu(y)$$

is well defined for a.e. $x \in X$. Let's estimate,

$$\|T_K f\|_2^2 = \int_X |\langle K_x, \bar{f}\rangle| \, d\mu(x)$$

where $K_x(y) = K(x,y)$.

$$\le \int_X \|K_x\|_2^2 \, \|f\|_2^2 \, d\mu(x)$$

$$= \|K\|_2^2 \cdot \|f\|_2^2$$

which shows that T_K defines a bounded operator on $L^2(X,\mu)$ with norm

$$\|T_K\| \le \|K\|_2 .$$

Let us compute the adjoint of T_K:

$$\langle T_K f, g\rangle = \int_X \overline{g(x)} \, T_K f(x) \, d\mu(x)$$

$$= \int_X \overline{g(x)} \int_X f(y) K(x,y) \, d\mu(y) \, d\mu(x)$$

$$= \int_X f(y) \overline{\int_X g(x) \overline{K(x,y)} \, d\mu(x)} \, d\mu(y)$$

$$= \langle f, T_{K^*} g \rangle$$

where $K^*(x, y) = \overline{K(y, x)}$.

Thus T_K is self-adjoint

$$\Longleftrightarrow K(x, y) = \overline{K(y, x)}$$

$$\forall x, y \in X.$$

II. Hahn-Banach and consequences.

Hahn-Banach refers in fact to a set of results that asserts the existence of continuous linear forms with additional properties. Whether hidden or explicit, convexity plays always a fondamental role.

We begin with a very general result called "Hahn-Banach, Analytic Form" that concerns $\mathbb{R}$-vector spaces.

Def. II. 1. A gauge on a $\mathbb{R}$-vector space V is a function $p : V \longrightarrow \mathbb{R}$ such that

(1) $p(\lambda \cdot x) = \lambda p(x)$, $\forall \lambda > 0$

(2) $p(x+y) \leq p(x) + p(y)$ $\forall x, y \in V$

Remark II.1: observe that $\forall r \in \mathbb{R}$ the sublevels $\{v \in V : p(v) < r\}$ and $\{v \in V : p(v) \leq r\}$ are convex.

Remark II.3: (fond. ex.) Let $C \subset V$ be a convex set with the property that $\forall v \in V \ \exists \lambda > 0$ with $v \in \lambda \cdot C$ (absorbant). Then

$$p(v) := \inf\{\lambda > 0 : v \in \lambda \cdot C\}$$

is a gauge function on V. In addition:

$$\{v : p(v) < 1\} \subset C \subset \{v : p(v) \leq 1\}.$$

<u>Theorem II.4</u> Let V be an $\mathbb{R}$-vector space, $p : V \to \mathbb{R}$ a gauge, $M \subset V$ an $\mathbb{R}$-vector subspace and $f : M \to \mathbb{R}$ a linear form with

$$f(v) \leq p(v) \quad \forall v \in M.$$

Then there exists $F : V \to \mathbb{R}$ linear extension of f with $F(v) \leq p(v)$ $\forall v \in V$.

<u>Remark</u>: this statement is often called "Hahn–Banach, the Analytic Form", perhaps because it contains an inequality (!?).

The proof uses Zorn's lemma which is a statement about (partially) ordered set and which we recall now.

Let $\mathcal{P}$ be a set with a partial order $\leq$.

A subset $Q \subset \mathcal{P}$ is totally ordered if $\forall a, b$ in in Q either $a \leq b$ or $b \leq a$. We say that $c \in P$ is an <u>upper bound</u> for a subset $Q \subset P$ if $a \leq c \quad \forall a \in Q$. We say that $m \in P$ is <u>maximal</u> if $m \leq x \Longrightarrow x = m$. Finally we say that P is <u>inductive</u> if every totally ordered subset $Q \subset P$ has an upper bound. We following is then Zorn's lemma which follows from the Axiom of Choice:

<u>Zorn's Lemma II.5</u> Let $P \neq \phi$ be ordered and inductive. Then P admits a maximal element.

With this we turn to the proof of Thm II.4.

Proof of Thm. II.4.

Let $\mathcal{P} = \{(h, D) : D \subset V$ is an $\mathbb{R}$-linear subspace, $M \subset D$, $h : D \to \mathbb{R}$ is linear, $h|_M = f$, $h(v) \leq p(v) \ \forall v \in D\}$.

We order $\mathcal{P}$ in the following way:

$$(h_1, D_1) \leq (h_2, D_2) \text{ if } D_1 \subset D_2$$

and $h_2|_{D_1} = h_1$. Clearly $\mathcal{P} \neq \phi$ since $(M, f) \in \mathcal{P}$.

Let's verify the hypotheses of Zorn's lemma.

Let $Q \subset \mathcal{P}$ be a totally ordered subset.

Define $\displaystyle E := \bigcup_{(D,h) \in Q} D$.

Since Q is totally ordered, E is a $\mathbb{R}$-vector subspace of V. Define $j : E \to \mathbb{R}$ by

$j|_D = h$ whenever $(h, D) \in \mathcal{Q}$.
Then one verifies easily that $(j, E) \in \mathcal{P}$; it is clearly an upper bound for $\mathcal{Q}$. By Zorn's lemma there is a maximal element $(F, E) \in \mathcal{P}$. All we have do show is that $E = V$.

Assume $E \neq V$, let $x_0 \notin E$ and $D := E + \mathbb{R}x_0$. Define $h: D \to \mathbb{R}$ by $h(v + t x_0) = F(v) + \alpha \cdot t$, $\forall v \in E$.
We are going to show that we can choose $\alpha \in \mathbb{R}$ in such a way that $h \leq p$ on D. Thus we need $\alpha \in \mathbb{R}$ such that

$$F(v) + t \cdot \alpha \leq p(v + t \cdot x_0) \quad \forall v \in E, \ \forall t \in \mathbb{R}.$$

Using the homogeneity of p, this amounts to the following inequalities:

(1) $F(x) + \alpha \leq p(x + x_0) \quad \forall x \in E$

(2) $F(x) - \alpha \leq p(x - x_0) \quad \forall x \in E.$

To show the existence of α we must show:

$$\sup_{y \in E} \{ f(y) - p(y - x_0) \} \leq \inf_{x \in E} \{ p(x + x_0) - F(x) \} \qquad (*)$$

or: $\forall x, y \in E$:

$$F(y) - p(y - x_0) \leq p(x + x_0) - F(x)$$

$$\Longleftrightarrow \quad F(x) + F(y) \leq p(x + x_0) + p(y - x_0)$$

<u>But:</u>

$$F(x) + F(y) = F(x+y) \cancel{= F((x+x_0)+(y-x_0))}$$

$$\leq p(x+y) \qquad \text{since } x+y \in E$$

$$= p((x + x_0) + (y - x_0))$$

$$\leq p(x + x_0) + p(y - x_0)$$

which shows $(*)$ and concludes the proof. $\boxed{\Xi}$