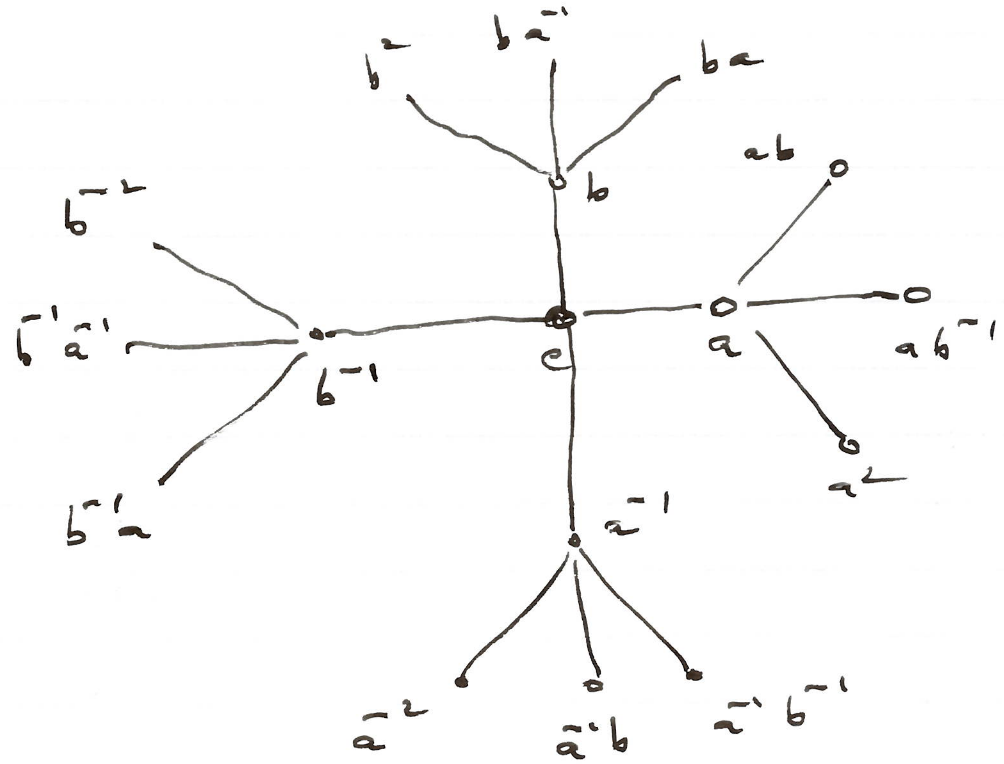
For those who know: let $\underset{\sim}{\mathbb{F}}(a,b)$ be the free group on two generators a, b and consider its Cayley graph relative to $\{a^{\pm 1}, b^{\pm 1}\}$:



Denote for a reduced word w, $S(w)$ the set of elements in $\underset{\sim}{\mathbb{F}}(a,b)$ whose reduced expression begins with w. Assume that $m: \mathcal{P}(\underset{\sim}{\mathbb{F}}(a,b)) \to [0,1]$ is a left invariant, finitely additive set functions.

Then: $$S(a) = S(ab) \sqcup S(ab^{-1}) \sqcup S(a^2)$$
$$= a\,S(b) \sqcup a\,S(b^{-1}) \sqcup a\,S(a)$$

which applying m and using invariance implies $m(S(b)) = m(S(b^{-1})) = 0$ and similarly $m(S(a)) = m(S(a^{-1})) = 0$.

Thus, for any subset $S \subset \mathbb{F}_2(a,b)$ we have $m(S) = m(S \cap \{e\})$. But this easily implies that $m \equiv 0$.

<u>Remark VI.24</u> (Exercise) The countable group Γ has property (F) if there exists a sequence $F_n \subset \Gamma$ of finite subsets such that $\forall \gamma \in \Gamma$:

$$\frac{|\gamma F_n \,\Delta\, F_n|}{|F_n|} \xrightarrow[n \to \infty]{} 0$$

Then one can show that the conclusion of Thm VI.20 holds for any countable group Γ satisfying (F). In particular such a Γ is amenable; it is a Thm of Følner that the converse holds.

Assume next that X is compact Hausdorff and $\varphi: X \longrightarrow X$ is a homeomorphism. We know now that there exist φ-invariant probability measures on X. The case where there is a unique such is particularly interesting:

<u>Thm VI.25</u> Assume that there is a unique φ-invariant probability measure μ on X. Then $\forall f \in C(X)$,

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(\varphi^k(x)) = \int_X f\,d\mu$$

the convergence being <u>uniform</u> in $x \in X$.

<u>Proof</u> :

(1) Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence in $\mathbb{N}_{\geq 1}$ and $(x_k)_{k\geq 1}$ a sequence in X. Define the sequence of probability measures,

$$\mu_k := \frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{\varphi^j(x_k)} \in M^1(X).$$

Let ν be any accumulation point of this sequence (in weak*-top.), that is:

$$\nu \in \bigcap_{N\geq 1} \overline{\{\mu_k : k \geq N\}} .$$

Then $\forall f \in C(X)$,

$$\mu_k(f\circ\psi) - \mu_k(f) = \frac{1}{n_k}\left\{ f(\psi^{n_k}(x_k)) - f(x_k) \right\}$$

thus $|\mu_k(f\circ\psi) - \mu_k(f)| \leq \dfrac{2\|f\|_\infty}{n_k}$

which implies that ν is ψ-invariant.

(2) If the convergence in the Thm is not uniform, there is $f \in C(X)$, $\varepsilon > 0$ ~~and~~ such that

$$\limsup_{n\to\infty} \sup_{x\in X} \left| \frac{1}{n}\sum_{k=0}^{n-1} f(\psi^k(x)) - \int f\,d\mu \right| > \varepsilon.$$

And hence there is a strictly increasing sequence $(n_k)_{k\geq 1}$ and a sequence $(x_k)_{k\geq 1}$ in X with

$$\left| \frac{1}{n_k}\sum_{j=0}^{n_k-1} f(\psi^j(x_k)) - \int f\,d\mu \right| > \varepsilon$$

which by (1) would lead to a ψ-invariant probability measure $\nu \neq \mu$. $\boxed{\xi}$

In specific situations it is relatively easy to establish uniqueness of the Ψ-invariant measure, as the following example shows:

Example VI.26 In the notations of Chapter II "The problem of measure", we defined the probability measure $\lambda \in M^1(\mathbb{R}/\mathbb{Z})$ which on continuous functions $f \in C(\mathbb{R}/\mathbb{Z})$ is given by

$$\int_{\mathbb{R}/\mathbb{Z}} f \, d\lambda = \int_0^1 f(\pi(x)) \, d\lambda(x)$$

where $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ is the canonical projection and "$d\lambda(x)$" the Lebesgue measure on $\mathbb{R}$ normalized so $\lambda([0,1]) = 1$. For $\alpha \in \mathbb{R}/\mathbb{Z}$ the map

$$T_\alpha : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$
$$x \longmapsto x + \alpha$$

is a homeomorphism. For $\alpha \in \mathbb{Q}/\mathbb{Z}$, let $\alpha = \frac{p}{q} + \mathbb{Z}$ where p, q are in $\mathbb{N}_{\geq 1}$ and coprime. Then $(T_\alpha)^q = id_{\mathbb{R}/\mathbb{Z}}$, and $\forall x \in \mathbb{R}/\mathbb{Z}$,

$$\mu_x := \frac{1}{q} \sum_{k=0}^{q-1} \delta_{T_\alpha^k(x)} \in M^1(\mathbb{R}/\mathbb{Z})$$

is an T_α-invariant probability measure; of course λ itself is T_α-invariant $\forall \alpha \in \mathbb{R}/\mathbb{Z}$. The point is then that for $\alpha \notin \mathbb{Q}/\mathbb{Z}$, λ is the <u>unique</u> T_α-invariant probability measure (exercise). As a result we get from Thm $\overline{\text{VI}}$.25 that

$$\frac{1}{n} \sum_{k=0}^{n-1} f(x + k \cdot \alpha) \longrightarrow \int_{\mathbb{R}/\mathbb{Z}} f \, d\lambda$$

uniformly, $\forall f \in C(\mathbb{R}/\mathbb{Z})$.

VI.3. Extreme points and the Krein-Milman Theorem.

Let E be an $\mathbb{R}$-vector space. For $x, y \in E$ we define

$$[x,y] = \{ tx + (1-t)y : t \in [0,1] \}$$

and

$$(x,y) = \{ tx + (1-t)y : t \in (0,1) \}$$

so for example $(x,x) = \{x\}$.

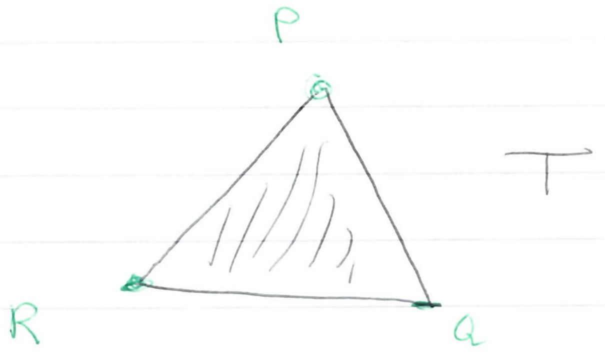
Def. VI.27 Let $A \subset E$ be a convex subset.

(1) $x \in A$ is an extreme point of A if $x \in (y,z)$ with $\{y,z\} \subset A$ implies

$$x = y = z.$$

(2) A convex subset $B \subset A$ is extrem in A if $(y,z) \cap B \neq \phi$ with $\{y,z\} \subset A$ implies

$$[y,z] \subset B.$$



Extreme points of T : $\{P\}, \{Q\}, \{R\}$

Extreme sets : T, $[P, Q]$, $[P, R]$, $[R, Q]$,
$\{P\}, \{Q\}, \{R\}$.

Denote by $ex(A)$ the subset of extreme points of a convex subset $A \subset E$.

<u>Thm VI.28</u> Let E be a TVS defined by a sufficient family of seminorms and $A \subset E$ convex, compact. Then

$$A = \overline{co(ex(A))}.$$

Before entering the proof, which uses the second geometric form of Hahn-Banach, we make the following

Remark VI.29: Let $A \subset B \subset C$ be convex subsets. Then if A is extreme in B and B is extreme in C, A is extreme in C.

Indeed, let $(y,z) \cap A \neq \phi$ with $\{y,z\} \subset C$. Since $A \subset B$ we conclude $(y,z) \cap B \neq \phi$ and since B is extreme in C, $[y,z] \subset B$ which implies since A is extreme in B that $[y,z] \subset A$.

Proof of Thm VI.28

(1) We show first that every closed convex extreme subset $B \subset A$ contains an extreme point of A. To this end, consider,

$$\mathcal{E}(B) := \left\{ C : C \subset B,\ C \text{ is closed convex extreme in } A \right\}$$

with the ordering $C_1 \leq C_2$ if $C_2 \subset C_1$.

This is a partially ordered set and we now show that every totally ordered subset has an upper bound. Let $\mathcal{C} \subset \mathcal{E}(B)$ be totally ordered, and define

$$M := \bigcap_{C \in \mathcal{C}} C .$$

Since $\mathcal{C}$ is totally ordered, given any finite subset $C_1, \ldots, C_n$ of $\mathcal{C}$ we may assume WLOG that $C_1 \subset \cdots \subset C_n$ and hence

$$\bigcap_{i=1}^{n} C_i = C_1 \neq \phi .$$

By compactness of A we deduce $M \neq \phi$.

Clearly M is closed, convex; in addition if $(y,z) \cap M \neq \phi$ with $\{y,z\} \subset A$ then $\forall\, C \in \mathcal{C}$ since C is extreme in A, $[y,z] \subset C$ and hence $[y,z] \subset M$. Thus $M \in \mathcal{E}(B)$ and is an upper bound for $\mathcal{C}$.

By Zorn's lemma there exists a maximal element $Z \in \mathcal{E}(B)$. We claim that Z is reduced to a point. Otherwise, assume $\{x,y\} \subset Z$ with $x \neq y$. By Hahn-Banach there is $F \in E^*$ with $F(x) < F(y)$.

Now consider $m := \max\{F(z) : z \in Z\}$ which exists since Z is compact, and

$$D = \{z \in Z : F(z) = m\}.$$

Then D is closed, convex; in addition

if $\{v, w\} \subset Z$ and $(v, w) \cap D \neq \phi$

then for some $t \in (0,1)$,

$$m = F(tv + (1-t)w) \geqslant F(v) \qquad (*)$$
$$\geqslant F(w).$$

From $t\,F(v) + (1-t)\,F(w) = F(tv + (1-t)w) \geqslant F(v)$

and ~~we get~~ $1-t > 0$ we get

$$F(w) \geqslant F(v)$$

and from $t\,F(v) + (1-t)\,F(w) \geqslant F(w)$

and $t > 0$ we get $F(v) \geqslant F(w)$.

Thus $F(v) = F(w) = m$ by $(*)$.

This shows that $[v, w] \subset D$ and hence

D is extreme in Z and hence in A by

Remark VI.25. On the other hand $F(x) < F(y)$

hence $x \notin D$ contradicting the maximality

of Z.

(2) In particular we now know that $ex(A) \neq \emptyset$.

Clearly, $A \supset \overline{Co(ex(A))}$. If now there were $x \in A$ and $x \notin \overline{Co(ex(A))}$ by the second geometric form of HB there is $\alpha \in \mathbb{R}$ and $F \in E^*$ such that

$$F(x) > \alpha > F(y) \quad \forall y \in \overline{Co(ex(A))}.$$

Consider then, as above,

$$m := \max\{F(z) : z \in A\}$$

and $\quad D := \{z \in A : F(z) = m\}$.

As above, D is closed, convex, extreme in A hence by (1) contains an extreme point e of A. But then:

$$F(e) \geq F(x) > \alpha > F(y) \quad \forall y \in ex(A),$$

a contradiction. ☒

Let $G \times X \to X$ be say a countable group acting by homeos on a compact Hausdorff space. An example is $G = \mathbb{Z}$ and the action is

$$\begin{aligned} \mathbb{Z} \times X &\to X \\ (m, x) &\mapsto \varphi^m(x) \end{aligned}$$

where $\varphi \in \mathrm{Homeo}(X)$. There is a measure theoretic notion of transitivity which plays a central role in dynamics:

<u>Def. VI. 30</u> A G-invariant probability measure $\mu \in M^1(X)$ is called <u>ergodic</u> if whenever $S \subset X$ is a G-invariant measurable subset, we have either $\mu(S) = 0$ or $\mu(X \setminus S) = 0$.

If now $M^1(X)^G$ denotes the subset of $M^1(X)$ of G-invariant probability measures then $M^1(X)^G$ is convex and weak*-closed, hence compact. We have:

__Lemma VI.31.__ $\mu \in M^1(X)^G$ is ergodic $\Leftrightarrow$ μ is an extreme point of $M^1(X)^G$.

__Proof:__ We only prove ($\Leftarrow$): if μ is not ergodic there is $S \subset X$ measurable G-invariant with $0 < \mu(S) < 1$.

Define then $\mu_1 := \frac{\mu|_S}{\mu(S)}$ and $\mu_2 := \frac{\mu|_{X \setminus S}}{\mu(X \setminus S)}$.

Then $\mu_1, \mu_2 \in M^1(X)^G$ and

$$\mu = \mu(S)\mu_1 + \mu(X \setminus S)\mu_2.$$

Since $\mu \neq \mu_1$, $\mu \neq \mu_2$, μ is not an extreme point. $\boxed{3}$

Krein-Milman then implies

<u>Corollary VI.32.</u> If there exists a G-inv. probability measure on X, then there is an ergodic one. In fact every G-invariant probability measure is a weak*-limit of convex combinations of ergodic ones.