Exercise Sheet 1

To be handed in until September 27

1. Lower bounds on the total absolute curvature

(a) Prove that a closed regular curve γ in $\mathbb{R}^2$ has

$$\int_\gamma |k|\, ds \geq 2\pi.$$

(b) Prove that a closed regular curve γ in $\mathbb{R}^3$ has

$$\int_\gamma |k|\, ds \geq \pi.$$

(c*) (Fenchel's Theorem) Any closed regular curve γ in $\mathbb{R}^3$ has

$$\int_\gamma |k|\, ds \geq 2\pi.$$

Moreover, there is an equality if and only if γ is a plane convex curve.

(d) Recall (Milnor) that a knotted regular curve in $\mathbb{R}^3$ has $\int_\gamma |k|\, ds \geq 4\pi$. Prove: This bound is sharp, i.e. this cannot be improved to

$$\int_\gamma |k|\, ds \geq a$$

for any $a > 4\pi$.

2. For those new to topology

(a) Let X be a topological space. Show that if X is path-connected, then X is connected.

(b) Show that the continuous image of a connected topological space is connected.

(c) Let $U \subseteq \mathbb{R}^n$ be open set. Show that if U is connected, then it is path-connected.

(d) Show that the continuous image of a compact topological space is compact.

(e) Let X be a compact and Hausdorff topological space. Show that a subset $A \subset X$ is compact iff it is closed.