Exercise Sheet 10

To be handed in until November 29

1. Optimal embedding of product of spheres

What is the lowest dimension d such that $S^m \times S^n$ embeds into $\mathbb{R}^d$?

2. Compact manifolds need at least one dimension higher to immerse

Prove that a compact n-dimensional manifold cannot be immersed into $\mathbb{R}^n$.

3. More about the tangent space

(a) Prove $\pi : TM \to M$ is a submersion.

(b) Show that TM is always orientable (even if M is not).

4. Orthogonal and unitary matrices as submanifolds

(a) Prove that $O(n)$ and $SO(n)$ are compact submanifold of $\mathbb{R}^{n\times n}$. Prove that $O(n)$ has two connected components.

(b) Prove that

$$U(n) := \{A \in \mathbb{C}^{n\times n} \mid \overline{A^T}A = I\},$$
$$SU(n) := \{A \in U(n) \mid \det U = 1\}$$

are both compact submanifolds of $\mathbb{C}^{n\times n} \cong \mathbb{R}^{2n^2}$.

(c) Compute the tangent spaces $T_I U(n)$ and $T_I SU(n)$ at the identity I.

(d) Are $U(n)$ and $SU(n)$ connected?

5. The complex projective space

Let $\mathbb{CP}^n := \{\text{complex lines in } \mathbb{C}^{n+1} \text{ through the origin}\}$. Define the function $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ by

$$z = (z^0, \dots, z^n) \mapsto [z] = \{\lambda z \,|\, \lambda \in \mathbb{C}\} \in \mathbb{CP}^n.$$

(a) Find coordinate charts that make $\mathbb{CP}^n$ into a smooth $2n$-manifold.

(b) Observe that $\mathbb{CP}^1 \cong S^2$.

(c) Let $S^{2n+1} := \{z \in \mathbb{C}^{n+1} \,|\, |z| = 1\}$. The map $h : S^{2n+1} \to \mathbb{CP}^n$ given by $h(z) := [z]$ is called the *Hopf fibration*. Prove that h is a submersion. The fibers $h^{-1}(q)$, $q \in \mathbb{CP}^n$, yield a decomposition of S^{2n+1} into circles.

(d) Observe that in the case $n = 1$ we get the classical Hopf fibration

$$h : S^3 \to S^2$$

as defined in exercise sheet 7.