Exercise Sheet 11

To be handed in until December 06

1. Some explicit computations of Lie brackets

Given $v \in \mathbb{R}^3$, define vector fields on $\mathbb{R}^3$ by

$$T_v(x) := v, \quad R_v(x) := v \times x, \quad x \in \mathbb{R}^3.$$

(a) Compute $[T_v, T_w]$, $[T_v, R_w]$, and $[R_v, R_w]$ for $v, w \in \mathbb{R}^3$.

(b) Write $R_i := R_{\frac{\partial}{\partial x^i}}$. Compute $[R_i, R_j]$.

2. Effect of product on Lie bracket

Let X, Y be differentiable vector fields and f, g differentiable functions on a manifold M. Prove that

$$[fX, gY] = fg[X, Y] + f(X \cdot g)Y - g(Y \cdot f)X.$$

3. Regularity of solutions

Let $X \in C^k(TM)$ and let $\gamma : (-T, T) \to M$ be a C^1 integral curve of X. Show that γ is C^{k+1}.

4. Closed sets can be obtained as the zero set of a smooth function and can be approximated from outside by regular open sets

(a) Show any closed set $A \subset \mathbb{R}^n$ is the zero set of some smooth function

$$f : \mathbb{R}^n \to \mathbb{R}.$$

(b) Let $A \subset \mathbb{R}^n$ be closed. Show there exist open sets $U_1 \supset U_2 \supset U_3 \supset \dots$ such that ∂U_j is a smooth $(n-1)$-manifold and

$$A = \bigcap_{j=1}^{\infty} U_j.$$

5. Covering groups

(a) Let G be a Lie group and K a discrete normal subgroup. The group homomorphism

$$G \to G/K$$

is a covering map. We call it a *covering homomorphism* and G a *covering group* of G/K. For an example see exercise 1 sheet 8.

Hint: Find an open neighborhood of the identity $e \in U$ such that $U \cdot U^{-1} \cap K = \{e\}$.

(b) A discrete normal subgroup K of a connected Lie group G lies in the center of G.

Hint: For $k \in K$ consider the map $g \mapsto gkg^{-1}$.

(c) Find a covering homomorphism

$$S^3 \times S^3 \to SO(4)$$

of degree 2. Since $S^3 \times S^3$ is simply-connected this shows that the universal covering group of $SO(4)$ is $S^3 \times S^3$.