Exercise Sheet 12

To be handed in until December 13

1. The tangent space of an implicitly defined submanifold

Let $f : M \to N$ be smooth and q a regular value of f. Set $P := f^{-1}(q)$. Prove:

$$T_pP = \ker(Df_p)$$

for all $p \in P$.

2. Groups covered by $SU(n)$

Let $n \geq 1$.

(a) Find the center of $SU(n)$.

(b) Show that $SU(n)$ covers only a finite number of groups.
(N.B. $SU(n)$ is simply-connected.)

3. Relation of flows and Lie brackets

We shall see later that ϕ^t_X commutes with ϕ^t_Y iff $[X, Y] = 0$.

(a) Given $v, w \in \mathbb{R}^3$, recall the two vector fields defined in exercise sheet 11 problem 1:

$$T_v(x) = v, \qquad R_w(x) = w \times x \qquad \text{for } x \in \mathbb{R}^3.$$

Describe the flows $\phi^t_{T_v}$, $\phi^t_{R_v}$ geometrically.

(b) Determine by geometric reasoning conditions on v, w such that the flows $\phi^t_{T_v}$, $\phi^t_{R_w}$ commute.

(c) Determine by computation conditions on v, w such that the Lie brackets $[T_v, R_w]$ vanishes.

4. Vector fields on matrix manifold

Fix a matrix $A \in \mathbb{R}^{n\times n}$. Define a vector field on $\mathbb{R}^{n\times n}$ by

$$X_A(B) = AB, \qquad \text{for } B \in \mathbb{R}^{n\times n}.$$

(a) Show that X_A is complete.

(b) Find a formula for the flow $\phi^t_{X_A}$.

5. An explicit example of a flow

Let X be the vector field on $\mathbb{R}^2$ given by $X(x,y) = (rx - x - y, ry + x - y)$ where $r = \sqrt{x^2+y^2}$.

(a) Draw the vector field X.

(b) Find the flow ϕ^t_X of X.

Hint: Work in polar coordinates.

(c) Draw the (open) domain of definition of ϕ_X.

(d) Is X complete? Restricted to which open subsets of $\mathbb{R}^2$ is the vector field X complete?