Exercise Sheet 13

To be handed in until December 20

1. Pushforward and pullback of vector fields

Let $\phi : M \to N$ be smooth, X a smooth vector field on M and Y a smooth vector field on N.

Define the *pushforward of* X *by* ϕ via

$$\phi_*(X)(q) := D_{\phi^{-1}(q)}\phi(X(\phi^{-1}(q))).$$

Define the *pullback of* Y *by* ϕ via

$$\phi^*(Y)(p) := (D_p\phi)^{-1}(Y(\phi(p))).$$

(a) Show that if ϕ is bijective, $\phi_*(X)$ is defined.

(b) Show that if ϕ is a diffeomorphism, $\phi_*(X) \in C^\infty(TN)$.

(c) Give an example where ϕ is bijective but $\phi_*(X)$ not smooth.

(d) Show that if ϕ is a local diffeomorphism, $\phi^*(Y)$ is defined and is in $C^\infty(TM)$.

(e) Suppose $\phi : M \to N$ and $\psi : N \to P$ are diffeomorphisms. Show

$$\phi^*\psi^* = (\psi \circ \phi)^*, \quad \psi_*\phi_* = (\psi \circ \phi)_*,$$

$$\phi^*\phi_* = id_{C^\infty(TM)}, \quad (\phi^{-1})^* = \phi_*.$$

2. Another flow example

Let $M = \mathbb{R}^2 \setminus \{0\}$ and consider the vector field $X(p) = \frac{\partial}{\partial x}$ for all $p \in M$. Define $b(p)$ to be the maximal (positive) time of existence for the flow of X starting at $p \in M$.

(a) Compute $b(p)$ for all $p \in M$.

(b) Verify that

$$b(p) = \liminf_{q \to p} b(q)$$

for all $p \in M$.

(c) Find the points where $\lim_{q\to p} b(q)$ does not exist.

(d) Verify directly that the maximal domain of existence

$$\mathcal{U} = \{(p,t) \,|\, x \in M, a(p) < t < b(p)\}$$

is open.

3. Left-invariant and right-invariant vector fields on matrix groups

Let $G = Gl(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$ be the (Lie) group of invertible matrices. For A in $\mathbb{R}^{n\times n}$ define vector fields $X_A, Y_A \in C^\infty(TG)$ by

$$X_A(B) := AB, \qquad Y_A(B) := BA$$

for $B \in G$.

For $C \in G$ define maps $L_C, R_C : G \to G$ by

$$L_C(B) := CB, \qquad R_C(B) := BC^{-1}$$

for $B \in G$. These maps are called *left and right translation by* C.

(a) Verify

$$L_C \circ L_D = L_{CD}, \qquad R_C \circ R_D = R_{CD}.$$

(b) Conclude that L, R are injective homomorphisms $L, R : G \to \mathrm{Diff}(G)$.

(c) We call a vector field $Z \in C^\infty(TG)$

- *left-invariant* if $L_C^*(Z) = Z$ for all $C \in G$,
- *right-invariant* if $R_C^*(Z) = Z$ for all $C \in G$.

Which of X_A, Y_A is left/right-invariant?

(d) Show that any left-invariant or right-invariant vector field on G is either of the form X_A or Y_A.

4. The Lie bracket and the matrix commutator

In this exercise we will show that

$$[Y_A, Y_B] = Y_{[A,B]}$$

where $[Y_A, Y_B]$ is calculated as the Lie bracket of vector fields and $[A,B]$ is the matrix commutator in $\mathbb{R}^{n\times n}$. In other words, the map $A \mapsto Y_A$, from matrices to vector fields, is a Lie algebra homomorphism.

(a) Show that the flow of Y_A is $\phi^t_{Y_A}(C) = Ce^{tA}$ for $C \in G = Gl(n,\mathbb{R})$.

(b) Show that the derivative of $\phi^t_{Y_A}$ at C is $D_C\phi^t_{Y_A}(E) = Ee^{tA}$ for $E \in \mathbb{R}^{n\times n}$.

(c) We will see next week that the Lie derivative agrees with the Lie bracket. Compute $[Y_A, Y_B]$ using that $[Y_A, Y_B](C) = \mathcal{L}_{Y_A} Y_B(C)$ for all $C \in G$.