Exercise Sheet 14

Not to be handed in

1. Another proof of the Jacobi identity

(a) Let X, Y, Z be smooth vector fields on a manifold. Prove the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

using the identity

$$\phi^*[Y, Z] = [\phi^* Y, \phi^* Z].$$

for $\phi = \phi^t$, the local flow of X, and then differentiating at $t = 0$.

(b) A derivation on an algebra $(A, \cdot)$ is a linear map $D : A \to A$ that satisfies the Leibniz rule:

$$D(y \cdot z) = (Dy) \cdot z + y \cdot (Dz).$$

Prove that the Lie derivative L_X is a derivation on the nonassociative algebra $(C^\infty(TM), [\cdot, \cdot])$.

2. Commutation error

Let ϕ_X^s, ϕ_Y^t be the flows of two vector fields X, Y. Show that the following formulas

(a) $\phi_Y^t \circ \phi_X^s(x) = x + sX + tY + O(|s|^2 + |t|^2)$

(b) $\phi_Y^t \circ \phi_X^s(x) - \phi_X^s \circ \phi_Y^t(x) = st[X, Y] + O(|s|^3 + |t|^3)$

(c) $\phi_Y^{-t} \circ \phi_X^{-s} \circ \phi_Y^t \circ \phi_X^s(x) = x + st[X, Y] + O(|s|^3 + |t|^3)$

hold in any coordinate system.

3. Old question, new computation

Given $v, w \in \mathbb{R}^3$, recall the vector fields defined in exercise sheet 11 problem 1:

$$T_v(x) = v, \qquad R_w(x) = w \times x \qquad \text{for } x \in \mathbb{R}^3.$$

We already computed the Lie brackets of these vector fields. Compute

$$L_{T_v} T_w, \quad L_{T_v} R_w, \quad L_{R_w} T_v, \quad L_{R_v} R_w$$

directly using the definition of the Lie derivative.

4. Twigs on a stream

Let X, Y be smooth vector fields. Consider the vector field

$$Y^t = (\phi_X^t)_*(Y).$$

(a) Consider the vector field

$$X(x,y) = \frac{\partial}{\partial x} - yg(x)\frac{\partial}{\partial y}$$

where $g : \mathbb{R} \to [0,1]$ is a bump function which is 1 on $[-1,1]$ and 0 outside $[-2,2]$. Draw the vector field X, and the flow ϕ_X^t by sketching some integral curves.

(b) Draw Y^t for $Y = \frac{\partial}{\partial y}$ for different times e.g. $t = 0, 1, 2, 5$.

(c) A metaphor for the vector field Y^t is "twigs on a stream". Explain. Is this a good metaphor?

5. Parking is difficult

A car moves in the plane $\mathbb{R}^2$, identified with $\mathbb{C}$. The movement of the car is given by its position $(x(t), y(t)) \in \mathbb{R}^2$ and its direction given by the unit vector $\theta \in S^1$. Moreover, we assume that the direction of movement always coincides with the direction of the car. Now consider the vector fields

$$X(x,y,\theta) := (\cos\theta, \sin\theta, 1)$$
$$Y(x,y,\theta) := (\cos\theta, \sin\theta, -1)$$

on the configuration space $M := \mathbb{R}^2 \times S^1$.

(a) What happens to the car if it moves by X? If it moves by Y?

(b) Compute $[X, Y]$.

(c) Why is parking so difficult? Explain fully and carefully.