Exercise Sheet 2

To be handed in until October 4

1. On curvature and torsion of curves in $\mathbb{R}^3$

(a) Compute the scalar curvature k and torsion l at $t = 0$ for the curve

$$t \mapsto (t, at^2, bt^3) \quad a, b \in \mathbb{R}.$$

(b) Show that if a curve γ in $\mathbb{R}^3$ has identically vanishing scalar torsion then γ lies in a plane.

(c) Suppose that a curve γ in $\mathbb{R}^3$ has constant scalar curvature and torsion. Show that γ must be a helix.

2. Curvature and torsion determine a curve in $\mathbb{R}^3$ up to rigid motion

Prove that any given smooth functions $k(s)$, $l(s)$, with $k(s) > 0$ determine a curve in $\mathbb{R}^3$ with curvature $k(s)$ and torsion $l(s)$ (where s is the arclength) that is unique up to rigid motion of space (i.e. a composition of rotations and translations).

Hint: Theorem. (*Existence and uniqueness for ODEs*)
Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be continuous. Moreover, suppose f is *locally Lipschitz in the second coordinate* i.e. for all $(t_0, y_0) \in U$ there is an open neighbourhood $W \subset U$ of (t_0, y_0) and $M > 0$ such that $|f(t, y_2) - f(t, y_2)| \leq M|y_2 - y_1|$ for all $(t, y_1), (t, y_2) \in W$.

For any $(t_0, y_0) \in U$ consider the ODE system

$$(*) = \begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(t_0) = x_0. \end{cases}$$

Then

i (Existence) There exists a small open interval I containing t_0 and a continuously differentiable function $y : I \to \mathbb{R}^n$ that solves $(*)$.

ii (Uniqueness) Suppose that there are two solutions $y, \tilde{y}$ of $(*)$ defined on intervals I and $\tilde{I}$ respectively. Then $y, \tilde{y}$ agree on the intersection $I \cap \tilde{I}$.

For the following problems, use the definitions:

$$\begin{aligned} k_1, k_2 &: \text{principal curvatures} \\ H = k_1 + k_2 &: \text{mean curvature} \\ K = k_1 k_2 &: \text{Gauss curvature} \end{aligned}$$

3. Curvatures of some standard surfaces

Compute the curvatures k_1, k_2, H and K for

(a) a sphere of radius R,

(b) a cylinder of radius R.

4. Curvatures of surfaces of revolution

A surface of revolution in $\mathbb{R}^3$ is defined by

$$M = M_f := \{(x, y, z) \in I \times \mathbb{R}^2 \,|\, f(x) = \sqrt{y^2 + z^2}\} \subset \mathbb{R}^3,$$

where $f : I \to \mathbb{R}$ is a smooth positive function, I an interval. The curve γ given by $y = f(x)$ in the plane $\mathbb{R}^2$ is called the *generator* of M. Find k_1, k_2, H and K for M.

Hint: You can use without proof (but think about it) that the principal directions of a surface of revolution are in the direction tangent to γ and normal to γ. Useful notation: $r = \sqrt{y^2 + z^2}$ and $re^{i\theta} = y + iz = (y, z)$.